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A Multidimensional Approach for Solving Multi-Objective Linear Programming Problems

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Abstract

Solving multi-objective linear programming problems (MOLPP) is a great challenge because it is essential in many real-life problems, especially manufacturing. Choosing the best solution is the goal of the decision-maker to produce a possibility to improve their ability to decide. Multi-dimensional scaling (MDS) gives this capability to make the right decision. In this study, we develop the MDS method for (MOLPP) in the work of Mrakhan et al. (2020). The method depends on embedding points in R^2. Start by building a matrix from a collection of points, and then use clustering to optimize the matrix dimensions and configure the points in R^2. The matrix has $(k_1 \times 2)$ dimensions, where k_1 is the big cluster of the points. Also, a center of points was used to find the scaling points, and then the center of the generated points was used to find a distance from the origin $(0,0)$. Our proposed algorithm offers a structured, efficient compromise solution for MOLPPs, outperforming traditional scalarization-based methods.

Keywords: Comprise solution; Multi-dimensional scaling; Multi-objective linear programming; Optimal advanced; Optimal average; Quadratic average.

Abstrak

Menyelesaikan masalah pemrograman linier multiobjektif (MOLPP) merupakan tantangan besar karena sangat penting dalam banyak masalah kehidupan nyata, terutama manufaktur. Memilih solusi terbaik adalah tujuan pembuat keputusan untuk menciptakan kemungkinan guna meningkatkan kemampuan mereka dalam mengambil keputusan. Penskalaan multidimensi (MDS) memberikan kemampuan ini untuk keputusan yang tepat. Pada studi ini, akan dikembangkan metode MDS untuk (MOLPP) dalam karya Mrakhan et al. (2020). Metode ini bergantung pada penyematan titik-titik di R^2: dimulai dengan membangun matriks dari kumpulan titik, lalu gunakan pengelompokan untuk mengoptimalkan dimensi matriks dan mengonfigurasi titik-titik di R^2. Matriks memiliki dimensi $(k_1 \times 2)$, dimana k_1 adalah klaster besar titik-titik. Selain itu, titik pusat digunakan untuk menemukan titik penskalaan, kemudian titik pusat tersebut digunakan untuk menemukan jarak dari titik asal $(0,0)$. Algoritma yang kami usulkan menawarkan solusi kompromi yang terstruktur dan efisien untuk MOLPP, yang mengungguli metode berbasis skalarisasi tradisional.

Kata Kunci: *Solusi terpadu; Skala multidimensi; Pemrograman linier multiobjektif; Lanjutan optimal; Rata-rata optimal; Rata-rata kuadratik.*

2020MSC: 90C29, 90C90.

1. INTRODUCTION

One of the most critical areas of optimization is multi-objective linear programming (MOLP), which deals with decision-making issues involving several frequently incompatible objectives [1]. The primary purpose is to find a solution that achieves these objectives. Efficiently offering a variety of

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Pareto-optimal alternatives to decision-makers. Numerous industries, including economics, engineering, environmental planning, logistics, and healthcare, have used MOLP substantially since its beginnings [1][2]. Although conventional approaches such as scalarization and transformation techniques have established groundwork for resolving these issues, difficulties still exist, especially when working with large-scale, dynamic, and uncertain systems. In recent years, especially from 2021 to 2024, there have been notable developments in MOLP's theoretical and computational facets.

Mrakhan et al. [3] presented a novel method to improve the accuracy and efficiency of solving MOLP problems. Likewise, Md. Abdul Alim and Marzia Yesmin [4] created a cutting-edge transformation methodology that enhanced the adaptability and scalability of current MOLP methods, highlighting its use intricate optimization scenarios, including network design and resource allocation. These pioneering investigations have added new approaches using probabilistic models and machine learning. The investigation of hybrid Bayesian optimization by Dogan and Prestwich [5] offers a strong framework for dealing with bi-level MOLP problems, especially when there is uncertainty and a hierarchical decision-making process. Q-learning and deep Q-networks are reinforcement learning approaches used to optimize goals dynamically. Their application in wind turbine management, where these techniques balance energy production with noise reduction, is a noteworthy example [6].

To improve the placement of healthcare facilities, for example, Davoodi and Calabrese [7] used bi-objective integer linear programming models, addressing essential societal requirements, including reducing commute time and guaranteeing task parity throughout institutions. Similarly, current evaluations of telecommunication network optimization emphasize the application of multi-objective algorithms to strike a balance between cost, latency, and reliability to guarantee peak performance in quickly changing digital infrastructures [8][9].

The improvement of methods has been another crucial area of advancement. At the same time, [10] and offered average approaches for tackling MOLP problems utilizing multiple mean strategies. Studies by Sulaiman and Mustafa [11] have used harmonic means to improve the previous methods. In this paper, we presented a novel method to improve the results of all the previous work using the multi-dimensional scaling (MDS) method.

2. METHODS

To illustrate the main ideas and concepts, we start with the following definitions:

Definition 1. [12] Let $R^n = \{x_1, \dots, x_n\}: x_j \in R$ for $j = \{1, \dots, n\}$. Then $\vec{x} = [\, x_1, \dots, x_n]^t$ is called a vector of n dimension.

Definition 2. [13] The numbers or values of the elements in a vector are called scalars.

Definition 3. [14] MDS is one of the dimensionality reduction techniques that convert multidimensional data into lower-dimensional space while keeping the intrinsic information.

Definition 4. [15] Cluster analysis is an exploratory data analysis tool for organizing observed data or cases into two or more groups.

2.1. Multi-objective Programming Problems Formulation

Subject to certain limitations, the mathematical model can be constructed in the following way to maximize (minimize) many objectives simultaneously:

$$\left.\begin{array}{l} Max.Z_i = c_i^t x + \beta_i \quad i = 1, \dots, r, \\ Min.Z_i = c_i^t x + \beta_i \quad i = r+1, \dots, r_1, \\ s.t. \\ X > 0, \end{array}\right\} \tag{1}$$

where X is vector of n-dimension, c is constant vector of n-dimension, β is constant vector of m-dimension, r is number of maximized objective functions, r_1 is number of maximized plus minimized objectives, $(r_1 - r)$ is number of minimized objectives, $c_i^t x + \beta_i \quad i = 1, \dots, r_1$ linear factors for feasible solutions, and $\beta_i \ (i = 1, \dots, r_1)$ are scalars. Since the objective functions with constraints are linear and all the variables are continuous, the problem is called a multi-objective linear programming problem (MOLPP) [16].

2.2. Classical Multidimensional Scaling (MDS)

The proximity between observations to visualize their spatial representation is called multidimensional scaling [17]. Start with an $(\mathrm{G} \times \mathrm{G})$ dissimilarity matrix D. To represent the G points in a low dimension, where the distances d_{ij} between them near the original δ_{ij} means $d_{ij} = \delta_{\mathrm{ij}}$ for all i, j [18]. One can formulate most MDS problems in terms of the optimization problem. A linear programming model for external analysis is presented [18][19]. Brusco proposed integer programming methods for the one-dimensional scaling of proximity matrices [20]. Laeuter and Ramadan [21] and [22] used optimization techniques to configure categorical data.

2.3. The Principal Concept

The first step in the strategy is to use the simplex method to maximize or minimize each goal function separately. The ideal values are

$$\begin{array}{l} Max\, Z_1 = \omega_1, \\ Max\, Z_2 = \omega_2, \\ \vdots \\ Max\, Z_r = \omega_r, \\ Min\, Z_{r+1} = \omega_{r+1}, \\ \vdots \\ Min\, Z_{r_1} = \omega_{r_1}. \end{array}$$

To form a single objective function and for maximum adding and for minimum subtracting of each result of dividing each z_i by ω_i, where $|\omega_{\mathrm{i}}| \neq 0_{\mathrm{i}}$, i.e., $Max\ Z = \sum_{i=1}^{r} \frac{Z_i}{|\omega_{\mathrm{i}}|} - \sum_{i=r+1}^{r_1} \frac{Z_i}{|\omega_{\mathrm{i}}|}$ [10] and subject to the same constraints in (1). All the presented methods aim to minimize the value of ω_i; different techniques were used to find a value from $\omega_i \ (i = 1, \dots, r_1)$ and denote this value by ρ. Note that as much as ω_{i} is small, the results are better. We have r_1 objective functions with a bounded feasible

region. To solve this problem, we used the optimal values of r_1 objective functions to find a new compromise objective function. This solution lies in the same feasible region of (1) [23].

The algorithm below characterizes the values of the objective functions individually, and then we construct ordered pairs that come from the Cartesian product (CP) of maximum and minimum values. Plot them in R^2. So, the points are visualized together in R^2. The big cluster means that the distances between the points are small. This gives the idea to choose it. Choosing a big cluster is reasonable for this purpose. Let the cluster contain g_1 points. Now, the dimension of D is $(g_1 \times 2)$, a matrix of the points of the cluster. Find the configuration points by MDS that are good approximations of the distances between the rows of D [18][19]. From these points, we find the center point, and then ρ is the distance between the origin and center points.

2.4. The Algorithm

Step 1: Plot the CP of individual values of $Max.Z_1$ and $Min.Z_1$,
Step 2: Choose the big cluster with g_1 elements. Construct a matrix D of dimension $(g_1 \times 2)$,
Step 3: Find a Euclidian distance $E = [d_{ij}], i, j = 1, \dots, g_1$,
Step 4: $A = -0.5 \times [d_{ij}^2]$,
Step 5: $B = \left(I - \frac{1}{g_1} J\right) \times A$, $\left(I - \frac{1}{g_1} J\right)$ where I is the identity matrix and J is the unit matrix, both of dimension $(g_1 \times g_1)$.
Step 6: λ_i and v_i are eigenvalues and eigenvectors, respectively, for B. Choose two largest eigenvalues.
Step 7: $S = (\sqrt{\lambda_1}\, v_1, \sqrt{\lambda_1}\, v_2)$, the coordinates of the points.
Step 8: Find the center point S^*, then ρ is the distance between $(0, 0)$ and S^*.
Note that we considered two eigenvalues to configure the points in R^2. For details of the algorithm, see [18].

3. RESULTS

To demonstrate the effectiveness and applicability of the proposed multidimensional approach in solving MOLPP, we present several numerical examples. These examples are solved using our method and are compared with results obtained using established methods in the literature.

Example 1: Solve the following (MOLPP)

$$\begin{aligned} Max.Z_1 &= 3x_1 + 2x_2 \\ Max.Z_2 &= 4x_1 + x_2 \\ Max.Z_3 &= 4x_1 - 2x_2 \\ Max.Z_4 &= 15x_1 + 4x_2 \\ Min.Z_5 &= -6x_1 + 2x_2 \\ Min.Z_6 &= -9x_1 + 3x_2 \\ Min.Z_7 &= -5x_1 + 2x_2 \end{aligned}$$

subject to

$$\begin{aligned} x_1 + x_2 &\leq 4 \\ x_1 - x_2 &\leq 2 \\ x_1, x_2 &\geq 0. \end{aligned}$$

Solution:

By the simplex method, the optimal values are $Z_1 = 11, Z_2 = 13, Z_3 = 10, Z_4 = 49, Z_5 = -16, Z_6 = -24$, and $Z_7 = -13$. So, the Cartesian product is the set $S = \{(11,-16),(11,-24),(11,-13),(13,-16),(13,-24),(13,-13),(10,-16),(10,-24),(10,-13),(49,-16),(49,-24),(49,-13)\}$. There are 12 elements. The plots are shown in Figure 1. From the graph below, we choose

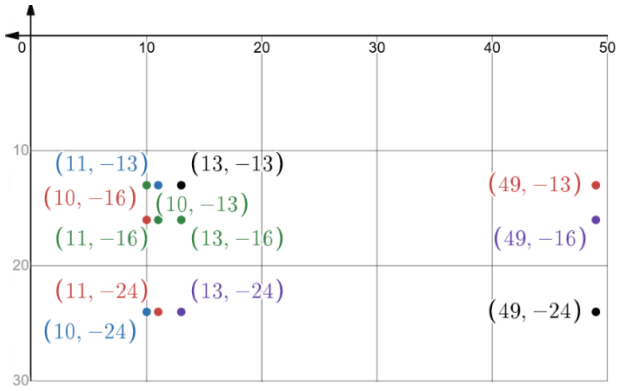

Figure 1. Cartesian products of example 1

The points in the big cluster are $(11,-16),(11,-24),(11,-13),(13,-16),(13,-24),(13,-13),(10,-16),(10,-24),(10,-13)$. So, the matrix D is

$$D = \begin{pmatrix} 11 & 11 & 11 & 13 & 13 & 13 & 10 & 10 & 10 \\ -16 & -24 & -13 & -16 & -24 & -13 & -16 & -24 & -13 \end{pmatrix}^t,$$

and

$$E = [d_{ij}] = \begin{pmatrix} 0 & 8 & 3 & 2 & 8.2 & 3.6 & 1 & 8 & 3.2 \\ 8 & 0 & 11 & 8.2 & 2 & 11.2 & 8 & 1 & 11 \\ 3 & 11 & 0 & 3.6 & 11.2 & 2 & 3.2 & 11 & 1 \\ 2 & 8.2 & 3.6 & 0 & 8 & 3 & 3 & 8.5 & 4.2 \\ 8.2 & 2 & 11.2 & 8 & 0 & 11 & 8.5 & 3 & 11.4 \\ 3.6 & 11.2 & 2 & 3 & 11 & 0 & 4.2 & 11.4 & 3 \\ 1 & 8 & 3.2 & 3 & 8.5 & 4.2 & 0 & 8 & 3 \\ 8 & 1 & 11 & 8.5 & 3 & 11.4 & 8 & 0 & 11 \\ 3.2 & 11 & 1 & 4.2 & 11.4 & 3 & 3 & 11 & 0 \end{pmatrix}.$$

Now, $A = -0.5 \times [d_{ij}^2]$

$$A = \begin{pmatrix} 0 & -32 & -4.5 & -2 & -34 & -6.5 & -0.5 & -32.5 & -5 \\ -32 & 0 & -60.5 & -34 & -2 & -62.5 & -32.5 & -0.5 & -61 \\ -4.5 & -60.5 & 0 & -6.5 & -62.5 & -2 & -5 & -61 & -0.5 \\ -2 & -34 & -6.5 & 0 & -32 & -4.5 & -4.5 & -36.5 & -9 \\ -34 & -2 & -62.5 & -32 & 0 & -60.5 & -36.5 & -4.5 & -65 \\ -6.5 & -62.5 & -2 & -4.5 & -60.5 & 0 & -9 & -65 & -4.5 \\ -0.5 & -32.5 & -5 & -4.5 & -36.5 & -9 & 0 & -32 & -4.5 \\ -32.5 & -0.5 & -61 & -36.5 & -4.5 & -65 & -32 & 0 & -60.5 \\ -5 & -61 & -0.5 & -9 & -65 & -4.5 & -4.5 & -60.5 & 0 \end{pmatrix},$$

and $B = \left(I - \frac{1}{9}J\right) \times A, \left(I - \frac{1}{9}J\right)$ which is

$$\begin{pmatrix} 2.8889 & -10.4444 & 7.8889 & 2.2222 & -11.1111 & 7.2222. & 3.2222 & -10.1111 & 8.2222 \\ -10.4444 & 40.2222 & -29.4444 & -11.1111 & 39.5556 & -30.1111 & -10.1111 & 40.5556 & -29.1111 \\ 7.8889 & -29.4444 & 21.8889 & 7.2222 & -30.1111 & 21.2222 & 8.2222 & -29.1111 & 22.2222 \\ 2.2222 & -11.1111 & 7.2222 & 5.5556 & -7.7778 & 10.5556 & 0.5556 & -12.7778 & 5.5556 \\ -11.1111 & 39.5556 & -30.1111 & -7.7778 & 42.8889 & -26.7778 & -12.7778 & 37.8889 & -31.7778 \\ 7.2222 & -30.1111 & 21.2222 & 10.5556 & -26.7778 & 24.5556 & 5.5556 & -31.7778 & 19.5556 \\ 3.2222 & -10.1111 & 8.2222 & 0.5556 & -12.7778 & 5.5556 & 4.5556 & -8.7778 & 9.5556 \\ -10.1111 & 40.5556 & -29.1111 & -12.7778 & 37.8889 & -31.7778 & -8.7778 & 41.8889 & -27.7778 \\ 8.2222 & -29.1111 & 22.2222 & 5.5556 & -31.7778 & 19.5556 & 9.5556 & -27.7778 & 23.5556 \end{pmatrix}$$

The largest eigenvalues of B are $\lambda_1 = 194,\ \lambda_2 = 14$, with the corresponding eigenvectors

$$V_1 = [0.1197 \quad -0.4547 \quad 0.3350\ 0.1197 \quad -0.4547 \quad 0.3350 \quad 0.1197 \quad -0.4547 \quad 0.3350]^{\mathrm{t}}$$
$$V_2 = [\ 0.0891 \quad 0.0891 \quad 0.0891 \quad -0.4454 \quad -0.4454 \quad -0.4454 \quad 0.3563 \quad 0.3563 \quad 0.3563],^{\mathrm{t}}$$

respectively. The coordinates of the points are

$$S = \left(\sqrt{194} \times \begin{bmatrix} 0.1197 \\ -0.4547 \\ 0.3350 \\ 0.1197 \\ -0.4547 \\ 0.3350 \\ 0.1197 \\ -0.4547 \\ 0.3350 \end{bmatrix}, \sqrt{14} \times \begin{bmatrix} 0.0891 \\ 0.0891 \\ 0.0891 \\ -0.4454 \\ -0.4454 \\ -0.4454 \\ 0.3563 \\ 0.3563 \\ 0.3563 \end{bmatrix} \right) = \begin{pmatrix} 1.667 & 0.333 \\ -6.333 & 0.333 \\ 4.666 & 0.333 \\ 1.667 & -1.666 \\ -6.333 & -1.666 \\ 4.666 & -1.666 \\ 1.667 & 1.333 \\ -6.333 & 1.333 \\ 4.666 & 1.333 \end{pmatrix}$$

So, we have 9 points in R^2, find the center point, say $S^* = (\bar{x}, \overline{y}) = \left(\frac{\sum_{i=1}^{9} x_i}{9}, \frac{\sum_{i=1}^{9} y_i}{9}\right) = (2.9000 \times 10^{-16}, 3.3333 \times 10^{-5})$, and the distance between S^* and the origin $(0,0)$ is $\rho = 0.00003$ which is our divided factor. Therefore

$$\mathrm{Z}^*_{\mathrm{optimal}} = 453333.3 \quad \text{at} \quad \mathrm{x}_1 = 3\,, \qquad \mathrm{x}_2 = 1.$$

Example 2: Solve the following (MOLPP)

$$Max.Z_1 = 4x_1 + 2x_2$$
$$Max.Z_2 = 3x_1 + 6x_2$$
$$Max.Z_3 = -8x_1 + 6x_2$$
$$Min.Z_4 = 5x_1 - 7x_2$$
$$Min.Z_5 = 2x_1 - 8x_2,$$

subject to

$$2x_1 + 6x_2 \leq 10$$
$$4x_1 - 2x_2 \leq 4$$
$$x_1, x_2 \geq 0.$$

Solution:
By the simplex method, the optimal values are $Z_1 = 8.5, Z_2 = 11.57,\ Z_3 = 10,\ Z_4 = -11.6,$ and $Z_5 = -13.3$. So, the Cartesian product is the set $S = \{(8.5, -11.6),\ (8.5, -13.3),\ (11.57, -11.62),$

$(11.57, -13.3), (10, -11.6), (10, -13.3)\}$. From Figure 2, we choose the big cluster.

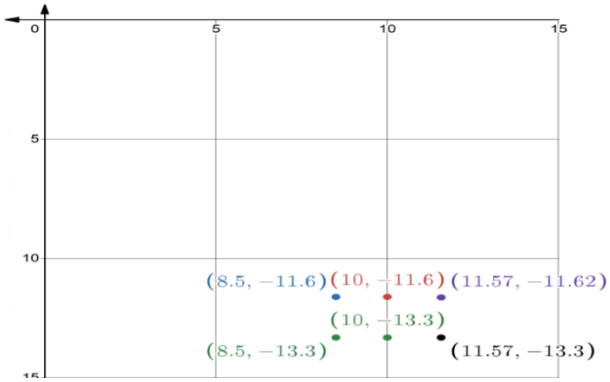

Figure 2. Cartesian products of example 2

The points in the big cluster are all the points: $\{(8.5, -11.6), (8.5, -13.3), (11.57, -11.62), (11.57, -13.3), (10, -11.6), (10, -13.3)\}$. The matrix D is

$$D = \begin{pmatrix} 8.5 & 8.5 & 11.57 & 11.57 & 10 & 10 \\ -11.6 & -13.3 & -11.62 & -13.3 & -11.6 & -13.3 \end{pmatrix}^{t},$$

and

$$E = [d_{ij}] = \begin{pmatrix} 0 & 1.73 & 3.09 & 4.8 & 1.5 & 3.2 \\ 1.73 & 0 & 4.78 & 3.07 & 3.23 & 1.53 \\ 3.09 & 4.78 & 0 & 1.71 & 1.59 & 3.25 \\ 4.8 & 3.07 & 1.71 & 0 & 3.3 & 1.6 \\ 1.5 & 3.23 & 1.59 & 3.3 & 0 & 1.7 \\ 3.2 & 1.53 & 3.25 & 1.6 & 1.7 & 0 \end{pmatrix}.$$

Now, $A = -0.5 \times [d_{ij}^2]$

$$A = \begin{pmatrix} 0 & -1.4965 & -4.774 & -11.52 & -1.125 & -5.12 \\ -1.4965 & 0 & -11.4242 & -4.7125 & -5.2165 & -1.1704 \\ -4.774 & -11.4242 & 0 & -1.4621 & -1.264 & -5.2813 \\ -11.52 & -4.7125 & -1.4621 & 0 & -5.4450 & -1.28 \\ -1.1250 & -5.2165 & -1.264 & -5.445 & 0 & -1.445 \\ -5.12 & -1.1704 & -5.2813 & -1.28 & -1.445 & 0 \end{pmatrix},$$

and $B = \left(I - \frac{1}{6}J\right) \times A, \left(I - \frac{1}{6}J\right) =$

$$\begin{pmatrix} 3.0659 & 1.5723 & -1.6280 & -3.1043 & 0.7809 & -0.6868 \\ 1.5723 & 3.0717 & -3.0869 & -1.6050 & -0.7127 & 0.7606 \\ -1.6280 & -3.0869 & 3.1034 & 1.6613 & 0.6920 & -0.7417 \\ -3.1043 & -1.6050 & 1.6613 & 3.1433 & -0.7843 & 0.6890 \\ 0.7809 & -0.7127 & 0.6920 & -0.7843 & 0.7459 & -0.7218 \\ -0.6868 & 0.7606 & -0.7417 & 0.6890 & -0.7218 & 0.7006 \end{pmatrix}.$$

The largest eigenvalues of B are $\lambda_1 = 9.4267$, $\lambda_2 = 4.4039$, with the corresponding eigenvectors

$$V_1 = [\ 0.4979\ \ 0.4944\ \ -0.5020\ \ -0.5055\ \ 0.0094\ \ 0.0059]^{t},$$

$$V_2 = [\,0.4068 \;\; -0.4175 \;\; 0.4065 \;\; -0.4083 \;\; 0.4113 \;\; -0.3987]^{\mathrm{t}}$$

respectively. The coordinates of the points are

$$S = \left(\sqrt{9.4267} \times \begin{bmatrix} 0.4979 \\ 0.4944 \\ -0.5020 \\ -0.5055 \\ 0.0094 \\ 0.0059 \end{bmatrix}, \sqrt{4.4039} \times \begin{bmatrix} 0.4068 \\ -0.4175 \\ 0.4065 \\ -0.4083 \\ 0.4113 \\ -0.3987 \end{bmatrix} \right) = \begin{pmatrix} 1.528 & 0.853 \\ 1.517 & -0.876 \\ -1.541 & 0.853 \\ -1.552 & -0.856 \\ 0.028 & 0.863 \\ 0.018 & -0.836 \end{pmatrix}.$$

So, we have 6 points in R^2, find the center point, $S^* = (-0.002, 0.001)$, and the distance between S^* and the origin $(0,0)$ is $\rho = 0.00022$, which is our divided factor. Therefore

$$\mathrm{Z}^*_{\mathrm{optimal}} = 142156.861 \quad \text{at } \mathrm{x}_1 = 0 \;, \mathrm{x}_2 = \frac{5}{3}.$$

In the following Table 1, we compare the results obtained by quadratic average, optimal advanced transformation, optimal average, multi-dimensional scaling.

Table 1. Comparison of different methods

Examples	Quadratic Average	Optimal Advanced Transformation	Optimal average	MDS
Example 1	$\mathrm{Z}^* = 6.04$	$\mathrm{Z}^* = 13.6$	$\mathrm{Z}^* = 11.82$	$\mathrm{Z}^* = 453333$
	$\mathrm{x}_1 = 3, \mathrm{x}_2 = 1$	$\mathrm{x}_1 = 3, \mathrm{x}_2 = 1$	$\mathrm{x}_1 = 3\,, \mathrm{x}_2 = 1$	$\mathrm{x}_1 = 3\,, \mathrm{x}_2 = 1$
Example 2	$\mathrm{Z}^* = 4.313$	$\mathrm{Z}^* = 5.685$	$\mathrm{Z}^* = 4.808$	$\mathrm{Z}^* = 142156$
	$\mathrm{x}_1 = 0\,, \mathrm{x}_2 = \frac{5}{3}$	$\mathrm{x}_1 = 0\,, \mathrm{x}_2 = \frac{5}{3}$	$\mathrm{x}_1 = 0,\ \mathrm{x}_2 = \frac{5}{3}$	$\mathrm{x}_1 = 0, \mathrm{x}_2 = \frac{5}{3}$

Notes:

1. If the objective functions are all in max case, then the Cartesian products will be in real line. In this case the problem is easier to solve.
2. As much as the cluster is big (contains most of the points), then the result is better. Otherwise, the $\mathrm{Z}^*_{\mathrm{optimal}}$ is small.

4. DISCUSSION

The proposed multidimensional scaling (MDS) method offers significant improvements over several prior approaches to solving multi-objective linear programming problems (MOLPP). For instance, compared to the transformation technique by [4] and [24], the MDS method integrates spatial and geometric information through clustering and scaling. This makes it more intuitive for decision-makers, especially in complex decision environments where multiple competing objectives must be reconciled visually and computationally.

Furthermore, while the average-based strategies proposed by [11][25] and improved by [10], they lack a graphical interpretation and do not capture the spatial relationships between feasible solutions. The MDS-based approach introduced in this study addresses this limitation by mapping solutions into a lower-dimensional Euclidean space, allowing the identification of central compromise points with preserved relative distances. This visualization capability makes the MDS approach more insightful, particularly when identifying Pareto-optimal clusters.

The algorithm also contrasts with the optimization-based methods by Mrakhan et al. [3] and Dogan and Prestwich [5], which focus on bilevel and probabilistic frameworks. While these methods

are effective in hierarchical or uncertain environments, they often require extensive computational resources and model-specific adjustments. In contrast, the MDS approach balances computational efficiency with solution interpretability. By incorporating eigenvalue decomposition and cluster analysis, it generates compromise solutions that not only satisfy the mathematical rigor but also enhance decision clarity across a wide range of practical applications.

5. CONCLUSIONS

We have studied MOLPP by using MDS. This method converts the cartesian products of the optimal values for the objective functions to the points in R^2. This method keeps the distances between the points. The results show a significant difference compared with other methods, as shown in Table 1.

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