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# Performance Parameters of the FM/FM/1 Queue with a Priori Impatience by the Soft Alpha-Cut Method

Mudimbiyi Ebondo Mbavu Durac[1]*, Mabela Makengo Matendo Rostin[2], and Tawaba Musian Ta – yen Gérard[3]
[1]Faculty of Economics and Management, University of Kabinda, Kabinda, D.R.Congo
[2]Department of Mathematics, Statistics and Computer Science, Faculty of Science and Technology, University of Kinshasa, Kinshasa, D.R.Congo
[3]Department of Mathematics, Statistics and Computer Science, Faculty of Science, National Pedagogical University, Kinshasa, D.R.Congo
Email: *duracmudimbiyi7@gmail.com

## Abstract

This study investigates the performance parameters of the fuzzy Markovian queueing system FM/FM/1 with a priori impatience using the soft alpha-cut method. The model incorporates the hesitation behavior of customers who decide whether to join a queue based on the expected waiting time. Unlike traditional parametric nonlinear programming approaches that combine multiple fuzzy arithmetics, the proposed method employs a single arithmetic framework based solely on alpha-cuts and interval operations, thereby simplifying and enhancing the computational efficiency. The study develops fuzzy formulations for key performance indicators, including server utilization rate, system throughput, average number of customers in the system and queue, and average waiting and residence times. A numerical application in a banking service context demonstrates the validity and practicality of the approach. The results show that each performance indicator is represented as a fuzzy number characterized by its support, mode, and membership function, allowing greater flexibility in managerial decision-making compared to the classical M/M/1 queueing system. Furthermore, the modal values of the fuzzy indicators coincide with the average values of their classical counterparts, confirming that the classical models are exceptional cases of the fuzzy ones.

**Keywords:** Fuzzy Markovian queue; A priori impatience, Soft alpha-cut method, Performance indicators, Queueing theory.

## Abstrak

*Penelitian ini menyelidiki parameter kinerja sistem antrean Markovian fuzzy FM/FM/1 dengan ketidaksabaran apriori menggunakan metode alpha-cut lunak. Model ini menggabungkan perilaku ragu-ragu pelanggan yang memutuskan untuk bergabung dalam antrean berdasarkan perkiraan waktu tunggu. Tidak seperti pendekatan pemrograman nonlinier parametrik tradisional yang menggabungkan beberapa aritmatika fuzzy, metode yang diusulkan menggunakan kerangka kerja aritmatika tunggal yang semata-mata didasarkan pada alpha-cut dan operasi interval, sehingga menyederhanakan dan meningkatkan efisiensi komputasi. Studi ini mengembangkan formulasi fuzzy untuk indikator kinerja utama, termasuk tingkat utilisasi server, throughput sistem, jumlah rata-rata pelanggan dalam sistem dan antrean, serta rata-rata waktu tunggu dan waktu tinggal. Aplikasi numerik dalam konteks layanan perbankan menunjukkan validitas dan kepraktisan pendekatan ini. Hasil penelitian menunjukkan bahwa setiap indikator kinerja direpresentasikan sebagai bilangan fuzzy yang dicirikan oleh fungsi pendukung, modus, dan keanggotaannya, yang memungkinkan fleksibilitas yang lebih besar dalam pengambilan keputusan manajerial dibandingkan dengan sistem antrean M/M/1 klasik. Lebih jauh lagi, nilai modal dari indikator fuzzy bertepatan dengan nilai rata-rata dari padanan klasiknya, yang mengonfirmasi bahwa model klasik adalah kasus luar biasa dari model fuzzy.*

***Kata Kunci:*** *Antrian Markov fuzzy; a priori impatience; metode soft alpha-cut; indikator kinerja; teori antrian.*

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*) Corresponding author
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## 1. INTRODUCTION

In this article, we analyze the performance of the FM/FM/1 Markovian queueing system by considering an important factor which is the a priori impatience by the soft alpha-cut method, in steady state, without losing sight of the fact that the methods of studying fuzzy Markovian queueing systems with impatient customers are similar to those of ordinary Markovian queueing systems with impatient customers.

To characterize the impatient behavior of customers, in the literature according to [1], there are three terminologies used in queuing theory, namely, hesitation otherwise known as a priori impatience, defined as the decision not to join the queue at all; denial otherwise known as a posteriori impatience, defined as joining a queue but leaving it without being served and recall or return or feedback, defined as the fact of a customer dissatisfied with the quality of the service and who decides to leave the queue to request or complete his service after a random time.

Despite many studies on queueing systems and the application of fuzzy theory to these systems in some cases, until now, the attention of some research has been focused on the time a customer must spend in a queue, while forgetting that people can take advantage of it to spend it in other services, this is how, through this study, we find a simple way to analyze the performance of such a system.

Regarding the priori impatience discussed in this article, the impatient customer must make the decision to join or not once arrived at the service stations when the service provider(s) are not inactive. The main factor that determines the customer's decision to join or not to join is the time before he/she gets the service [2] and [3]. However, the customer always makes his/her decision based on the length of the queue because time is invisible. This work is not the first to deal with a priori impatience, in the literature there are the works of pioneers such as the work of [4], [5], [6], [7] and many others.

In the literature many approaches have been developed, the most widely used of which is that of parametric nonlinear programs which combine two fuzzy arithmetic's to achieve the desired results: arithmetic based on the Zadeh extension principle and that of alpha – cuts and intervals. In this article, we use a new scientific approach, the soft alpha – cut approach, which differs from that of parametric nonlinear programs in that it only uses a single arithmetic called alpha – cuts and intervals instead of two. For this new approach, almost all calculations are non – fuzzy and are based only on interval arithmetic. The fact that these are non –fuzzy makes the approach soft compared to the approach of parametric non – linear programs.

The queuing model suggested by this paper is applicable to a variety of fields, including manufacturing, production, communication, transportation and networking, restaurants, bank or retail counters, ATMs, parks and others.

This article sets itself the objective are (1) to check if the mathematical approach of soft alpha – cuts can facilitate the analysis of the performance of the fuzzy Markovian queue FM/FM/1 with a priori impatience; and (2) to evaluate the performance indicators allowing the optimization of the management of the said system while reducing waits in the queue and the costs which may result there from. This evaluation will be done solely from a mathematical aspect.

The rest of this article is organized as follows: Section 2 presents some definitions necessary for the arithmetic of alpha-cuts and intervals and reviews the soft alpha-cut method. Section 3 presents the results while analyzing the performance of the FM/FM/1 queue with a priori impatience.

Section 4 provides the discussions based on the results of the proposed numerical application in the banking field. Section 5 ends the study with a conclusion.

## 2. DEFINITIONS

### 2.1. Fuzzy subset

**Definition 1**. [8][9] Let $X$ be a reference set or a universe. A fuzzy subset $\tilde{A}$ (Fussy set) of $X$ is defined by the membership function $\mu_{\tilde{A}}(x) : X \to [0,1]$. We denote

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x) \;/\; x \in X)\}. \tag{1}$$

**Definition 2.** [10][11] Let us consider $\tilde{A}$ a fuzzy subset in the universe $X$. The support $Supp(\tilde{A})$, the height $h(\tilde{A})$, the core $Noy(\tilde{A})$ and the alpha – cut $\tilde{A}_\alpha$ are numbers defined respectively $\forall\, \alpha \in [0\ ,\ 1]$ by:

$$Supp(\tilde{A}) = \{x \in X \;/\; \mu_{\tilde{A}}(x) > 0\}, \tag{2}$$
$$h(\tilde{A}) = Sup\{\, \mu_{\tilde{A}}(x) \;/\; x \in X\}, \tag{3}$$
$$Noy(\tilde{A}) = \{x \in A / \mu_{\tilde{A}}(x) = 1\}, \tag{4}$$
$$\tilde{A}_\alpha = \{x \in X \;/\; \mu_{\tilde{A}}(x) \geq \alpha\}. \tag{5}$$

According to [12], the membership function of a fuzzy subset $\tilde{A}$ can be expressed in terms of the characteristic functions of its alpha-cuts by the relation

$$\mu_{\tilde{A}}(x) = \operatorname*{Sup}_{\alpha \in [0\,,1]} \min\{\alpha, \mu_{\tilde{A}_\alpha}(x)\}, \tag{6}$$

where $\mu_{\tilde{A}_\alpha}(x) = \begin{cases} 1 & if \quad x \in \tilde{A}_\alpha \\ 0 & otherwise \end{cases}$.

**Definition 3.** [13][14][15] A fuzzy subset $\tilde{A}$ is said to be normal or normalized if and only if $\mathrm{h}(\widetilde{\mathrm{A}}) = 1$ and is convex if and only if

$$\forall x_1, x_2 \in X \; et \; \lambda \in [0; 1],\;\; \mu_{\tilde{A}}[\lambda x_1 + (1-\lambda)x_2] \geq min[\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2)]. \tag{7}$$

### 2.2. Fuzzy number

**Definition 4.** [10] A fuzzy subset $\tilde{A}$ defined on the universe $\mathbb{R}$ of real numbers is a fuzzy number if its membership function $\mu_{\tilde{A}} : \mathbb{R} \longrightarrow [0,1]$ satisfies the following conditions:
1. $\mu_{\tilde{A}}$ is convex;
2. $\mu_{\tilde{A}}$ is normal, that is, there exists an $x \in \mathbb{R}$ such that $\mu_{\tilde{A}}(x) = 1$;
3. $\tilde{A}$ is upper semi continuous;
4. $Supp(\tilde{A})$ is bounded in $\mathbb{R}$ .

**Definition 5.** [16] A fuzzy number $\tilde{A}$ is said to be positive (resp. negative) if and only if $\mu_{\tilde{A}}(x) = 0$, $\forall\, x < 0$ ($\mu_{\tilde{A}}(x) = 0, \forall x > 0$). This is denoted $\widetilde{\mathrm{A}} > 0$ (resp. $\widetilde{\mathrm{A}} < 0$);

If $\tilde{A}$ is a fuzzy interval, any real number m such that $\mu_{\tilde{A}}(m) = 1$ is said to be a modal value or mode or even a mean value of $\tilde{A}$. In this case, $Noy(\tilde{A})$ is the set of modal values of $\tilde{A}$.

**Definition 6.** [12] Let $\tilde{A}$ and $\tilde{B}$ be two fuzzy numbers. $\tilde{A}$ is strictly less than $\tilde{B}$ if and only if

$$\forall\, x \in supp(\tilde{A}), \forall\, y \in supp(\tilde{B}); \qquad x < y.$$

We denote

$$\tilde{A} < \tilde{B} \Leftrightarrow sup\{supp(\tilde{A})\} < inf\{supp(\tilde{B})\}. \tag{8}$$

**Definition 7.** [17][18] A fuzzy number $\tilde{A}$, with membership function $\mu_{\tilde{A}}$, is said to be triangular, if there exist three real numbers $a < b < c$ such that:

$$\mu_{\tilde{A}}(x) = \begin{cases} (x-a)/(b-a) & if \quad a \le x \le b \\ (c-x)/(c-b) & if \quad b < x \le c \\ 0 & otherwise \end{cases} \tag{9}$$

where $b$ is the unique modal value of $\tilde{A}$ ; that is, $\mu_{\tilde{A}}(b) = 1$.

## 2.3. Arithmetic on fuzzy numbers

### 2.3.1. Arithmetic on Intervals

Consider two ordinary closed intervals $[a_1, a_2]$ and $[b_1, b_2]$ of $\mathbb{R}$. We define the arithmetic operation $*$ on these two intervals by the following relation:

$$[a_1 \ ; \ a_2] * [b_1 \ ; \ b_2] = \{a * b / a_1 \le a \le a_2 \ \ et \ \ b_1 \le b \le b_2\}, \tag{10}$$

where division is possible only if zero does not belong to $[b_1, b_2]$ [13]. In the case of the operations $+;\ -;\ \times;\ \div$; the equality (10) simplifies as follows:

$$[a_1 \ ; \ a_2] + [b_1 \ ; \ b_2] = [a_1 + b_1;\ a_2 + b_2\ ], \tag{11}$$

$$[a_1 \ ; \ a_2] - [b_1 \ ; \ b_2] = [a_1 - b_2;\ a_2 - b_1], \tag{12}$$

$$[a_1 \ ; \ a_2] \times [b_1 \ ; \ b_2] = [min\, P;\ max\, P], \tag{13}$$

where $P = (a_1 b_1;\ a_1 b_2;\ a_2 b_1\ ;\ a_2 b_2)$.

$$\frac{[a_1 \ ; \ a_2]}{[b_1 \ ; \ b_2]} = [min\, H\, ; max\, H] \text{ where } H = \left(\frac{a_1}{b_1};\ \frac{a_1}{b_2}; \frac{a_2}{b_1};\ \frac{a_2}{b_2}\right). \tag{14}$$

Furthermore, if the above intervals are defined in the set of positive real numbers $\mathbb{R}^+$, the multiplication, division and inverse interval operations can still be written as follows:

$$[a_1\ ;\ a_2] \times [b_1\ ;\ b_2] = [a_1 b_1\ ;\ a_2 b_2], \tag{15}$$

$$\frac{[a_1 \ ; \ a_2]}{[b_1 \ ; \ b_2]} = \left[\frac{a_1}{b_2};\ \frac{a_2}{b_1}\right], \tag{16}$$

$$[a_1\ ;\ a_2]^{-1} = \left[\frac{1}{b_2};\ \frac{1}{b_1}\right]. \tag{17}$$

### 2.3.2. Arithmetic of alpha-cuts

Let us consider $\tilde{A}$ and $\tilde{B}$ two fuzzy subsets of respective α-cuts $\tilde{A}_\alpha = [A_\alpha^L, A_\alpha^U]$ and $\tilde{B}_\alpha = [B_\alpha^L, B_\alpha^U]$; with $\alpha \in [0\ ;\ 1]$. The four operations $\oplus$; $\ominus$; $\odot$ and $\oslash$ are performed on $\tilde{A}_\alpha$ and $\tilde{B}_\alpha$, according to [11] and [19], by passing to their α-cuts in the following manner:

$$\left(\tilde{A} \oplus \tilde{B}\right)_\alpha = \tilde{A}_\alpha + \tilde{B}_\alpha = [A_\alpha^L, A_\alpha^U] + [B_\alpha^L, B_\alpha^U], \tag{18}$$

$$\left(\tilde{A} \ominus \tilde{B}\right)_\alpha = \tilde{A}_\alpha - \tilde{B}_\alpha = [A_\alpha^L, A_\alpha^U] - [B_\alpha^L, B_\alpha^U], \tag{19}$$

$$\left(\tilde{A} \odot \tilde{B}\right)_\alpha = \tilde{A}_\alpha \cdot \tilde{B}_\alpha = [A_\alpha^L, A_\alpha^U] \cdot [B_\alpha^L, B_\alpha^U], \tag{20}$$

$$\left(\tilde{A} \oslash \tilde{B}\right)_\alpha = \frac{\tilde{A}_\alpha}{\tilde{B}_\alpha} = \frac{[A_\alpha^L, A_\alpha^U]}{[B_\alpha^L, B_\alpha^U]}. \tag{21}$$

### 2.3.3. Alpha-cut and interval arithmetic

**Definition 8.** [20] Performing fuzzy arithmetic by alpha-cut and interval arithmetic involves successively using:

- Relations (11); (12); (13) and (14) for defuzzification;
- Relations (18); (19); (20) and (21) for ordinary calculations on closed real intervals;
- Relation (6) for fuzzification.

## 2.4. SOFT ALPHA-CUT METHOD

Let us consider a fuzzy queueing system whose parameters are fuzzy numbers $\tilde{\xi}_1;\ \tilde{\xi}_2;\ \cdots;\tilde{\xi}_n$ and let us denote the performance indicator that we wish to calculate by $\tilde{\psi}$. In the classical model, let us assume that these parameters and this indicator are respectively denoted by $\xi_1\,;\xi_2;\ \ldots;\xi_n$ and $\psi$. In the classical model, the formula of $\psi$ is often given by

$$\psi = f(\xi_1\,;\xi_2;\ \ldots;\xi_n), \tag{22}$$

where $f$ is a function with $n$ real variables defined using the fundamental operations $\ll +;\ -;\div;\times \gg$ in $\mathbb{R}$; while in the fuzzy model, this same formula is written as:

$$\tilde{\psi} = \tilde{f}\left(\tilde{\xi}_1;\ \tilde{\xi}_2;\ \cdots;\tilde{\xi}_n\right), \tag{23}$$

where $\tilde{f}$ is a function with n fuzzy variables, defined using the fuzzy operations $\ll \oplus; \ominus; \odot; \oslash \gg$ in $\tilde{\mathcal{F}}(\mathbb{R})$, where $\tilde{\mathcal{F}}(\mathbb{R})$ represents the set of fuzzy numbers. The following procedure allow to calculate any performance indicator.

**Step 1**: Determine the general expressions of the alpha – cuts of $\tilde{\psi}$. This requires that we determine the alpha – cuts of the fuzzy numbers $\tilde{\xi}_1;\ \tilde{\xi}_2;\ \cdots;\tilde{\xi}_n$, denoted respectively by $\tilde{\xi}_{1\alpha};\ \tilde{\xi}_{2\alpha};\ \cdots;\tilde{\xi}_{n\alpha}$. In this case, equality (23) becomes:

$$\tilde{\psi}_\alpha = \tilde{f}\left(\tilde{\xi}_{1\alpha};\ \tilde{\xi}_{2\alpha};\ \cdots;\tilde{\xi}_{n\alpha}\right). \tag{24}$$

**Step 2**: In (24), each $\tilde{\xi}_{i\alpha}$ (with $i = 1; 2;\ \ldots; n$) is a closed interval of $\mathbb{R}$ whose bounds are real functions of variables $\alpha$. In other words, each $\tilde{\xi}_{i\alpha}$ is written as:

$$\tilde{\xi}_{i\alpha} = [\varphi_i(\alpha);\ \phi_i(\alpha)], \tag{25}$$

where $\varphi_i$ and $\phi_i$ are real functions of variables $\alpha$ $(1 \le i \le n)$. Substituting (25) into (24), we obtain: $\tilde{\psi}_\alpha = \tilde{f}([\varphi_1(\alpha); \phi_1(\alpha)];\ [\varphi_2(\alpha);\ \phi_2(\alpha)];\ \ldots;\ [\varphi_n(\alpha);\ \phi_n(\alpha)])$. (26)

**Step 3**: Perform the arithmetic operations contained in (26) using the "alpha arithmetic – cuts and intervals". These operations require solving at least two PNLPs for each multiplication and for each division of the intervals found in (26).

**Step 4**: The final cut obtained is of the form:

$$\tilde{\psi}_\alpha = \left[\tilde{\psi}^L_\alpha;\ \tilde{\psi}^U_\alpha\right],\ 0 \le \alpha \le 1, \tag{27}$$

where $\tilde{\psi}^L$ and $\tilde{\psi}^U$ are real functions of variable $\alpha$.

Thus, we have just found the alpha-cut sought throughout this process. This means that the performance indicator $\tilde{\psi}$ is a fuzzy number, whose general expression of the alpha-cut is that found in (27). Let us remember that, when a fuzzy queueing system is characterized by a fuzzy performance indicator as in (23), the soft alpha-cut method leads us to classical queueing systems whose non-fuzzy parameters are in (27). In short, the soft alpha-cut method transforms a fuzzy queueing system into a family of classical queueing systems using the arithmetic of alpha-cuts and intervals where all the calculations are almost non-fuzzy.

**Step 5**: Fuzzify the result found in (27) by constructing the membership function of $\tilde{\psi}$. This function is defined by the reciprocals of the functions $\tilde{\psi}^L$ and $\tilde{\psi}^U$ of the alpha–cut in (27):

$$\mu_{\tilde{\psi}}(x) = \begin{cases} \left(\tilde{\psi}^L\right)^{-1}(x) & if \quad \tilde{\psi}^L(0) \le x \le \tilde{\psi}^L(1), \\ \left(\tilde{\psi}^U\right)^{-1}(x) & if \quad \tilde{\psi}^U(1) < x \le \tilde{\psi}^U(0), \\ 0 & otherwise. \end{cases} \tag{28}$$

**Step 6**: The support and the modal value of the fuzzy number $\tilde{\psi}$ are given by the alpha–cut of $\tilde{\psi}$ in (27) whose:

- The support of $\tilde{\psi}$, is the zero-level cut (i.e. $\alpha = 0$), denoted :

$$\tilde{\psi}_0 = \left]\tilde{\psi}^L(0);\ \tilde{\psi}^U(0)\right[. \tag{29}$$

- The modal value (or mode) of $\tilde{\psi}$ is defined for $\alpha = 1$ by:

$$\tilde{\psi}_1 = \tilde{\psi}^L(1) = \tilde{\psi}^U(1). \tag{30}$$

## 3. RESULTS

### 3.1. Performance of The FM/FM/1 System with a Priori Impatience

#### 3.1.1. System description and Assumptions

The FM/FM/1 queueing system with a priori impatience is a single-server fuzzy Markovian queueing system. It is modeled by a homogeneous life and death process (LDP) with fuzzy birth rate $\tilde{\lambda}_k$ and fuzzy death rate $\tilde{\mu}_k$ defined respectively by

$$\tilde{\lambda}_k = \tilde{\lambda}\,;\ \tilde{\lambda} > 0 \quad and \quad \tilde{\mu}_k = k\tilde{\mu} \text{ for } k = 0,1,\ldots.$$

The FM/FM/1 queueing system with a priori impatience has the same characteristics as the M/M/1 model. However, the state probabilities $p_k$ are replaced by the fuzzy state probabilities $\tilde{p}_k$.

### 3.1.2. Steady-state analysis

**Theorem 1.** Let $\tilde{p}_k$ be the fuzzy state probabilities, in steady state, of the FM/FM/1 system with a priori impatience, with respective rates $\tilde{\lambda}_k = \tilde{\lambda}$ and $\tilde{\mu}_k = k\tilde{\mu}$ for $k = 0,1, \ldots$ Then

$$\tilde{p}_k = \frac{\tilde{\rho}^k}{k!} e^{-\tilde{\rho}} \text{ for } k = 0,1, \ldots \text{ and } \tilde{p}_0 = e^{-\tilde{\rho}},$$

$$\tilde{\rho} = \frac{\tilde{\lambda}}{\tilde{\mu}} \text{ is the fuzzy traffic intensity.}$$

**Proof:**

The Kolmogorov differential equations of process of life and death in steady state are given by:

$$\begin{cases} 0 = \lambda_{k-1} p_{k-1} - (\lambda_k + \mu_k) p_k + \mu_{k+1} p_{k+1}, \\ 0 = -\lambda_0 p_0 + \mu_1 p_1. \end{cases}$$

By replacing the classical parameters of the equations by the fuzzy parameters, we obtain:

$$\begin{cases} 0 = \tilde{\lambda}_{k-1} \tilde{p}_{k-1} - (\tilde{\lambda}_k + \tilde{\mu}_k) \tilde{p}_k + \tilde{\mu}_{k+1} \tilde{p}_{k+1}, & (31) \\ 0 = -\tilde{\lambda}_0 \tilde{p}_0 + \tilde{\mu}_1 \tilde{p}_1. & (32) \end{cases}$$

Replacing $\tilde{\lambda}_k$ and $\tilde{\mu}_k$ by their values in (31) and (32), we obtain the following system of equations:

$$\begin{cases} 0 = \tilde{\lambda} \tilde{p}_{k-1} - (\tilde{\lambda} + k\tilde{\mu}) \tilde{p}_k + (k+1) \tilde{\mu} \tilde{p}_{k+1}, & (33) \\ 0 = \tilde{\mu} \tilde{p}_1 - \tilde{\lambda} \tilde{p}_0. & (34) \end{cases}$$

Eq. (33) and (34) determine a recurrent system by which:

$$\tilde{p}_1 = \frac{\tilde{\lambda}}{\tilde{\mu}} \tilde{p}_0, \tag{35}$$

$$\tilde{p}_2 = \frac{1}{2} \left( \frac{\tilde{\lambda}}{\tilde{\mu}} \right)^2 \tilde{p}_0, \tag{36}$$

$$\vdots$$

$$\tilde{p}_k = \frac{1}{k!} \left( \frac{\tilde{\lambda}}{\tilde{\mu}} \right)^k \tilde{p}_0. \tag{37}$$

Let

$$\tilde{p}_k = \frac{\tilde{\rho}^k}{k!} \tilde{p}_0 \text{ with } \tilde{\rho} = \frac{\tilde{\lambda}}{\tilde{\mu}}. \tag{38}$$

At this level, let us introduce the normalization condition $\sum_{k=0}^{+\infty} p_k = 1$ and this condition associated with the relation (38), gives us:

$$\sum_{k=0}^{+\infty} \frac{\tilde{\rho}^k}{k!} \tilde{p}_0 = 1 \implies \tilde{p}_0 = \frac{1}{\sum_{k=0}^{+\infty} \frac{\tilde{\rho}^k}{k!}} = \frac{1}{e^{\tilde{\rho}}} = e^{-\tilde{\rho}}.$$

Thus,

$$\tilde{p}_0 = e^{-\tilde{\rho}}. \tag{39}$$

From where

$$\tilde{p}_k = \frac{\tilde{\rho}^k}{k!} e^{-\tilde{\rho}} \text{ for } k = 0,1, \ldots. \quad \blacksquare \tag{40}$$

**Remark 1**: If $\rho < 1$, then $\frac{\tilde{\lambda}}{\tilde{\mu}} < 1, i.e. \tilde{\lambda} < \tilde{\mu}$. We have, $Noy(\tilde{\lambda}) < Noy(\tilde{\mu})$: Fuzzy condition of ergodicity or stability.

### 3.2. Performance indicators of the FM/FM/1 queueing system with a priori impatience

The performance indicators of the system are only determined from the fuzzy state probabilities $\tilde{p}_k$ of (39) and (40) in the case where the system is ergodic or stable.

**Proposition 1.** ( Fuzzy server utilization rate: $\tilde{\mathcal{U}}$) Let $\tilde{\mathcal{U}}$ be the fuzzy server utilization rate. Then

$$\tilde{\mathcal{U}} = \frac{\tilde{\mu}-\tilde{\mu}e^{-\tilde{\rho}}}{\tilde{\mu}}. \tag{41}$$

**Proof:**

Since the fuzzy server utilization rate is defined as the proportion of time during which the server is busy over a time interval, we then have: $\tilde{\mathcal{U}} = 1 - \tilde{p}_0$, or according to (39), $\tilde{p}_0 = e^{-\tilde{\rho}}$ which gives: $\tilde{\mathcal{U}} = 1 - e^{-\tilde{\rho}}$. Composing by $\tilde{\mu}/\tilde{\mu}$, , we obtain,

$$\tilde{\mathcal{U}} = \left(1 - e^{-\tilde{\rho}}\right)\frac{\tilde{\mu}}{\tilde{\mu}} = \frac{\tilde{\mu}-\tilde{\mu}e^{-\tilde{\rho}}}{\tilde{\mu}} \text{ with } \tilde{\rho} = \frac{\tilde{\lambda}}{\tilde{\mu}} \text{ fuzzy traffic intensity.}$$

From where $\tilde{\mathcal{U}} = \frac{\tilde{\mu}-\tilde{\mu}e^{-\tilde{\rho}}}{\tilde{\mu}}$ . ■

**Proposition 2**. (Fuzzy flow rate of the system: $\tilde{d}$) Let $\tilde{d}$ be the fuzzy flow rate of the system. Then;

$$\tilde{d} = \tilde{\mu} - \tilde{\mu}e^{-\tilde{\rho}}. \tag{42}$$

**Proof**:

Generally, the flow rate of a system is calculated either by the input into the system (input flow rate $\tilde{d}_e$) or by the output of the system (output flow rate $\tilde{d}_s$). In the case of stable systems, these two flow rates are equal and we have: $\tilde{d} = \tilde{d}_e = \tilde{d}_s$. This is also the case with the system under study, for which the service is performed with a rate $\tilde{\mu}$ in each state where the system contains at least one customer, we therefore have:

$$\tilde{d} = \left(\tilde{\mathbb{P}}[non - empty\ queue]\right) \times \tilde{\mu} = \sum_{k=0}^{+\infty} \tilde{p}_k \times \tilde{\mu} = \left(1 - e^{-\tilde{\rho}}\right) \times \tilde{\mu} = \tilde{\mu} - \tilde{\mu}e^{-\tilde{\rho}},$$

with $\tilde{\mathbb{P}}$ fuzzy probability. From where $\tilde{d} = \tilde{\mu} - \tilde{\mu}e^{-\tilde{\rho}}$. ■

**Proposition 3.** (Fuzzy average number of customers in the system: $\tilde{N}_S$) Let $\tilde{N}_S$ be the fuzzy average number of customers in the system. Then,

$$\tilde{N}_S = \tilde{\rho}. \tag{43}$$

**Proof**:

As the fuzzy process $\{\tilde{X}(t) : t \geq 0\}$ denotes the fuzzy number of customers in the system at a date t, the fuzzy average number of customers in the system in steady state, is the fuzzy mathematical expectation of the fuzzy random variable $\tilde{X}$. We have:

$$\widetilde{N}_S = \tilde{E}(\tilde{X}) = \tilde{\mathbb{P}}[\tilde{X} \leq k] = \sum_{k=0}^{+\infty} k \times \tilde{p}_k = \sum_{k=0}^{+\infty} k \times \frac{\tilde{\rho}^k}{k!} \times e^{-\tilde{\rho}} = e^{-\tilde{\rho}} \sum_{k=0}^{+\infty} k \times \frac{\tilde{\rho}^k}{k!}$$
$$= e^{-\tilde{\rho}} \sum_{k=0}^{+\infty} \frac{\tilde{\rho}^k}{(k-1)!} = e^{-\tilde{\rho}} \times \tilde{\rho} \times e^{\tilde{\rho}} = \tilde{\rho}.$$

Therefore $$\widetilde{N}_S = \tilde{\rho}.$$ ■

**Proposition 4**. (Fuzzy average number of customers in the queue: $\widetilde{N}_f$) Let $\widetilde{N}_f$ be the average number of customers in the queue. Then,

$$\widetilde{N}_f = \frac{\tilde{\lambda} + \tilde{\mu} e^{-\tilde{\rho}} - \tilde{\mu}}{\tilde{\mu}}. \tag{44}$$

**Proof**:

Since the system under study is single-server, the presence of $k$ customers in the system implies that there are $k-1$ in the queue. So $\widetilde{N}_f$, is therefore the theoretical fuzzy mean of the fuzzy random variable $\tilde{X}$ , with $\tilde{X}$ taking all values between 0 and $k-1$.

$$\widetilde{N}_f = \tilde{E}(\tilde{X}) = \tilde{\mathbb{P}}[\tilde{X} \leq k-1] = \sum_{k=1}^{+\infty} (k-1) \times \tilde{p}_k = \sum_{k=1}^{+\infty} k\, \tilde{p}_k - \sum_{k=1}^{+\infty} \tilde{p}_k$$
$$= \sum_{k=0}^{+\infty} k\, \tilde{p}_k - \left(\sum_{k=0}^{+\infty} \tilde{p}_k - \tilde{p}_0\right).$$

By introducing the fuzzy normalization condition $\sum_{k=1}^{+\infty} \tilde{p}_k = 1$ and the relation $\sum_{k=0}^{+\infty} k \times \tilde{p}_k$ i.e. (43) into this relation, we find:

$$\widetilde{N}_f = \widetilde{N}_S - (1 - \tilde{p}_0) = \tilde{\rho} - \left(1 - e^{-\tilde{\rho}}\right) = \tilde{\rho} + e^{-\tilde{\rho}} - 1 = \frac{\tilde{\lambda}}{\tilde{\mu}} + e^{-\tilde{\rho}} - 1 = \frac{\tilde{\lambda} + \tilde{\mu} e^{-\tilde{\rho}} - \tilde{\mu}}{\tilde{\mu}}.$$

Hence $$\widetilde{N}_f = \frac{\tilde{\lambda} + \tilde{\mu} e^{-\tilde{\rho}} - \tilde{\mu}}{\tilde{\mu}}.$$ ■

**Proposition 5**. (Fuzzy average residence time: $\tilde{t}_S$) Let $\tilde{t}_S$ be the fuzzy average residence time. Then

$$\tilde{t}_S = \frac{\tilde{\lambda}}{\tilde{\mu}^2 - \tilde{\mu}^2 e^{-\tilde{\rho}}}. \tag{45}$$

**Proof**:

Little's fuzzy formula is given by:

$$\widetilde{N}_S = \tilde{t}_S \times \tilde{d}. \tag{46}$$

From (46), we draw $\tilde{t}_S$, we have:

$$\tilde{t}_S = \frac{\widetilde{N}_S}{\tilde{d}} = \frac{\tilde{\rho}}{\tilde{\mu}(1 - e^{-\tilde{\rho}})} = \frac{\frac{\tilde{\lambda}}{\tilde{\mu}}}{\tilde{\mu}(1 - e^{-\tilde{\rho}})} = \frac{\tilde{\lambda}}{\tilde{\mu}^2(1 - e^{-\tilde{\rho}})} = \frac{\tilde{\lambda}}{\tilde{\mu}^2 - \tilde{\mu}^2 e^{-\tilde{\rho}}}.$$

Hence $$\tilde{t}_S = \frac{\tilde{\lambda}}{\tilde{\mu}^2 - \tilde{\mu}^2 e^{-\tilde{\rho}}}.$$ ■

**Proposition 6.** (Fuzzy average time in the queue: $\tilde{t}_f$) Let $\tilde{t}_f$ be the fuzzy average time of customers waiting in the queue. Then,

$$\tilde{t}_f = \frac{\tilde{\lambda} + \tilde{\mu} e^{-\tilde{\rho}} - \tilde{\mu}}{\tilde{\mu}^2 - \tilde{\mu}^2 e^{-\tilde{\rho}}}. \tag{47}$$

**Proof**:

The average fuzzy time spent in the server by each client is $1/\tilde{\mu}$ , it is easy to conclude that the average fuzzy time spent in the FM/FM/1 queue is:

$$\tilde{t}_f = \tilde{t}_S - \frac{1}{\tilde{\mu}} = \frac{\tilde{\lambda}}{\tilde{\mu}^2 - \tilde{\mu}^2 e^{-\tilde{\rho}}} - \frac{1}{\tilde{\mu}} = \frac{\tilde{\lambda} - (\tilde{\mu}^2 - \tilde{\mu}^2 e^{-\tilde{\rho}})}{\tilde{\mu}^2 - \tilde{\mu}^2 e^{-\tilde{\rho}}} = \frac{\tilde{\lambda} + \tilde{\mu} e^{-\tilde{\rho}} - \tilde{\mu}}{\tilde{\mu}^2 - \tilde{\mu}^2 e^{-\tilde{\rho}}}.$$

Therefore
$$\tilde{t}_f = \frac{\tilde{\lambda} + \tilde{\mu} e^{-\tilde{\rho}} - \tilde{\mu}}{\tilde{\mu}^2 - \tilde{\mu}^2 e^{-\tilde{\rho}}}.$$
■

### 3.3. Numerical Application

**Problem:**

In a one-stop commercial bank with infinite capacity and exponential arrival and service times. The average time between arrivals is about 10 minutes and the average time per service is about 8 minutes. The operation of the bank is subject to a fuzzy queueing system with a priori impatience.

**Questions:**

1) Show that the system is ergodic;
2) Determine, in steady state, the fuzzy performance indicators of this system.

#### 3.3.1. Solution

By hypothesis, this is a fuzzy Markovian queueing system with a priori impatience, of the form FM/FM/1 with a single server and infinite capacity. In a classical model, the arrival rate $\lambda = 1/10$ minute, or $\lambda = 60/10 = 6$ arrivals per hour and the service rate $\mu = 1/8$ minute, or $\mu = 60/8 = 7.5$ departures per hour.

Since the arrival rate is about 10 minutes, or about 6 arrivals per hour, and the average rate per service is about 8 minutes, or about 7.5 departures per hour, this suggests that these are fuzzy rates. Assuming that these fuzzy rates $\tilde{\lambda}$ and $\tilde{\mu}$ are triangular fuzzy numbers, we can write:

$$\tilde{\lambda} = (5/\,6/\,7\,) \text{ and } \tilde{\mu} = (6.5/\,7.5/\,8.5).$$

Thus, in a fuzzy model, where the rates $\tilde{\lambda}$ and $\tilde{\mu}$ are fuzzy variables, the performance indicators $\tilde{\mathcal{U}}$; $\tilde{d}$; $\tilde{N}_S$; $\tilde{N}_f$; $\tilde{t}_S$ and $\tilde{t}_f$ also become fuzzy numbers [see formulas (41) – (45) and (47)].

#### 3.3.2. Resolution by the soft alpha-cut method

The arithmetic of alpha-cuts and intervals can be easily applied, since the rates $\tilde{\lambda}$ and $\tilde{\mu}$ are triangular fuzzy numbers. For the computation of the fuzzy numbers $\tilde{\mathcal{U}}$; $\tilde{d}$; $\tilde{N}_S$; $\tilde{N}_f$; $\tilde{t}_S$ and $\tilde{t}_f$, the soft alpha-cut method first requires the determination of the alpha-cuts. These cuts are classical expressions, for which all non-fuzzy computation techniques are applied.

Let's start by showing that the system is ergodic; we have $Noy(\tilde{\lambda}) < Noy(\tilde{\mu})$ i.e. $6 < 7.5$, so the system is ergodic and it's possible to calculate the performance indicators. Thus, to determine $\tilde{\mathcal{U}}$; $\tilde{d}$; $\tilde{N}_S$; $\tilde{N}_f$; $\tilde{t}_S$ and $\tilde{t}_f$, it suffices to:

1) Write equalities (41) – (45) and (47) in alpha - cuts:

$$\tilde{U}_\alpha = \frac{\tilde{\mu}_\alpha - \tilde{\mu}_\alpha e^{-\tilde{\rho}_\alpha}}{\tilde{\mu}_\alpha} \tag{48}$$

$$\tilde{d}_\alpha = \tilde{\mu}_\alpha - \tilde{\mu}_\alpha e^{-\tilde{\rho}_\alpha} \tag{49}$$

$$\left(\tilde{N}_S\right)_\alpha = \tilde{\rho}_\alpha \tag{50}$$

$$\left(\tilde{N}_f\right)_\alpha = \frac{\tilde{\lambda}_\alpha + \tilde{\mu}_\alpha e^{-\tilde{\rho}_\alpha} - \tilde{\mu}_\alpha}{\tilde{\mu}_\alpha} \tag{51}$$

$$(\tilde{t}_S)_\alpha = \frac{\tilde{\lambda}_\alpha}{\tilde{\mu}_\alpha^2 - \tilde{\mu}_\alpha^2 e^{-\tilde{\rho}_\alpha}} \tag{52}$$

$$\left(\tilde{t}_f\right)_\alpha = \frac{\tilde{\lambda}_\alpha + \tilde{\mu}_\alpha e^{-\tilde{\rho}_\alpha} - \tilde{\mu}_\alpha}{\tilde{\mu}_\alpha^2 - \tilde{\mu}_\alpha^2 e^{-\tilde{\rho}_\alpha}} \tag{53}$$

2) Calculation of the alpha - cuts $\tilde{\lambda}_\alpha$ and $\tilde{\mu}_\alpha$. From relation (10) we obtain :

$$\tilde{\lambda}_\alpha = [\alpha + 5;\ -\alpha + 7], \tag{54}$$

$$\tilde{\mu}_\alpha = [\alpha + 6.5;\ -\alpha + 8.5]. \tag{55}$$

3) Let's use the expressions from (11) to (14) of interval arithmetic to obtain the alpha - cuts of these parameters using eq. (54) and (55) in (48) – (53) as follows:

$$\tilde{\rho}_\alpha = \frac{\tilde{\lambda}_\alpha}{\tilde{\mu}_\alpha} = \frac{[\alpha+5;\ -\alpha+7]}{[\alpha+6.5;\ -\alpha+8.5]} = [min\, P(\alpha);\ \ max\, P(\alpha)],$$

where $min\, P(\alpha)$ and $max\, P(\alpha)$ are determined by solving two parametric nonlinear program (PNLPs).

Hence, $P(\alpha) = \{p_1(\alpha);\ p_2(\alpha);\ p_3(\alpha);\ p_4(\alpha)\}$; with

$$p_1(\alpha) = \frac{\alpha+5}{\alpha+6.5}\ ;\ \ p_2(\alpha) = \frac{\alpha+5}{-\alpha+8.5}\ ;\ p_3(\alpha) = \frac{-\alpha+7}{\alpha+6.5}\ ;\ p_4(\alpha) = \frac{-\alpha+7}{-\alpha+8.5}.$$

Solving these two PNLPs gives:

$$min\, P(\alpha) = p_2(\alpha) = \frac{\alpha+5}{-\alpha+8.5} \qquad and \quad max\, P(\alpha) = p_3(\alpha) = \frac{-\alpha+7}{\alpha+6.5}.$$

Thus,

$$\tilde{\rho}_\alpha = \left[\frac{\alpha+5}{-\alpha+8.5}\ ;\ \frac{-\alpha+7}{\alpha+6.5}\right] \tag{56}$$

If $\alpha = 0$, then $Supp(\tilde{\rho}) = \left]\frac{5}{8.5}\ ;\ \frac{7}{6.5}\right[ = ]0.824\ ;\ \ 1.077[$.
If $\alpha = 1$, then $Noy(\tilde{\rho}) = \frac{6}{7.5} = 0.8$.

$\tilde{\rho}$ as a triangular fuzzy number can then be written $\tilde{\rho} = (0.5\ /\ 0.8\ /\ 1.1)$.

Let's start by calculating the remarkable terms in (48); (49); (50); (51); (52) and (53). Indeed,

$$\tilde{\rho}_\alpha = \left[\frac{\alpha+5}{-\alpha+8.5}\ ;\ \frac{-\alpha+7}{\alpha+6.5}\right].$$

The modal value of $\tilde{\rho}$ being 0.8, then:

$$*\ e^{-\tilde{\rho}_\alpha} = e^{-0.8} = 0.45, \tag{57}$$

$$*\ \tilde{\mu}_\alpha e^{-\tilde{\rho}_\alpha} = [\alpha + 6.5;\ -\alpha + 8.5] \times 0.45 = [0.45\alpha + 2.925\ ;\ -0.45\alpha + 3.825], \tag{58}$$

$* \ \tilde{\mu}_\alpha^2 = \tilde{\mu}_\alpha \times \tilde{\mu}_\alpha = [\alpha + 6.5;\ -\alpha + 8.5] \times [\alpha + 6.5;\ -\alpha + 8.5] = [min\, Q(\alpha);\ \ max\, Q(\alpha)],$

where $min\, Q(\alpha)$ and $max\, Q(\alpha)$ are determined by solving two PNLPs. Hence, $Q(\alpha) = \{q_1(\alpha);\ q_2(\alpha);\ q_3(\alpha);\ q_4(\alpha)\}$ with $q_1(\alpha) = (\alpha + 6.5)(\alpha + 6.5)$ ; $q_2(\alpha) = (\alpha + 6.5)(-\alpha + 8.5)$ ; $q_3(\alpha) = (-\alpha + 8.5)(\alpha + 6.5)$ ; and $q_4(\alpha) = (-\alpha + 8.5)(-\alpha + 8.5)$. Solving these two PNLPs gives:

$$min\, Q(\alpha) = q_1(\alpha) = (\alpha + 6.5)^2 \text{ and } max\, Q(\alpha) = q_4(\alpha) = (-\alpha + 8.5)^2.$$

Hence $\tilde{\mu}_\alpha^2 = [(\alpha + 6.5)^2\ ;\ (-\alpha + 8.5)^2]$.

Let

$$\tilde{\mu}_\alpha^2 = [\alpha^2 + 13\alpha + 42.25\ ; \alpha^2 - 17\alpha + 72.25] \tag{59}$$

$$* \ \tilde{\mu}_\alpha^2 e^{-\tilde{\rho}_\alpha} = [\alpha^2 + 13\alpha + 42.25\ ; \alpha^2 - 17\alpha + 72.25] \times 0.45$$
$$= [0.45\alpha^2 + 5.85\alpha + 19.0125\ ;\ 0.45\alpha^2 - 7.65\alpha + 32.5125], \tag{60}$$

$$* \ \tilde{\mu}_\alpha - \tilde{\mu}_\alpha e^{-\tilde{\rho}_\alpha} = [\alpha + 6.5;\ -\alpha + 8.5] - [0.45\alpha + 2.925\ ;\ -0.45\alpha + 3.825]$$
$$= [1.45\alpha + 2.675\ ;\ -1.45\alpha + 5.575], \tag{61}$$

$$* \ \tilde{\mu}_\alpha e^{-\tilde{\rho}_\alpha} - \tilde{\mu}_\alpha = [0.45\alpha + 2.925\ ;\ -0.45\alpha + 3.825] - [\alpha + 6.5;\ -\alpha + 8.5]$$
$$= [1.45\alpha - 5.575\ ;\ -1.45\alpha - 2.675], \tag{62}$$

$$* \ \tilde{\mu}_\alpha^2 - \ \tilde{\mu}_\alpha^2 e^{-\tilde{\rho}_\alpha} = [\alpha^2 + 13\alpha + 42.25\ ; \alpha^2 - 17\alpha + 72.25]$$
$$-[0.45\alpha^2 + 5.85\alpha + 19.0125\ ;\ 0.45\alpha^2 - 7.65\alpha + 32.5125]$$
$$= [0.55\alpha^2 + 20.65\alpha + 9.7375\ ;\ 0.55\alpha^2 - 22.85\alpha + 53.2375], \tag{63}$$

$$* \ \tilde{\lambda}_\alpha + \tilde{\mu}_\alpha e^{-\tilde{\rho}_\alpha} - \tilde{\mu}_\alpha \ = [\alpha + 5;\ -\alpha + 7] + [1.45\alpha - 5.575\ ;\ -1.45\alpha - 2.675]$$
$$= [2.45\alpha - 0.575\ ;\ -2.45\alpha - 4.325]. \tag{64}$$

Let's calculate the alpha - cuts of $\tilde{\mathcal{U}}$; $\tilde{d}$; $\tilde{N}_S$; $\tilde{N}_f$; $\tilde{t}_S$ and $\tilde{t}_f$ .

**(i) Calculation of $\tilde{\mathcal{U}}_\alpha$**

Eq. (55) and (61) in (48), we can write:

$$\tilde{\mathcal{U}}_\alpha = \frac{[1.45\alpha + 2.675\ ;\ -1.45\alpha + 5.575]}{[\alpha + 6.5;\ -\alpha + 8.5]} = [min\, R(\alpha);\ \ max\, R(\alpha)],$$

with, $R(\alpha) = \{r_1(\alpha);\ r_2(\alpha);\ r_3(\alpha);\ r_4(\alpha)\}$; such that

$$r_1(\alpha) = \frac{1.45\alpha + 2.675}{\alpha + 6.5}\ ; \quad r_2(\alpha) = \frac{1.45\alpha + 2.675}{-\alpha + 8.5}$$
$$r_3(\alpha) = \frac{-1.45\alpha + 5.575}{\alpha + 6.5}\ ;\ \ r_4(\alpha) = \frac{-1.45\alpha + 5.575}{-\alpha + 8.5}$$

Solving these two PNLPs, we obtain:

$$min\, R(\alpha) = r_2(\alpha) = \frac{1.45\alpha + 2.675}{-\alpha + 8.5} \text{ and } max\, R(\alpha) = r_3(\alpha) = \frac{-1.45\alpha + 5.575}{\alpha + 6.5}.$$

Hence,

$$\tilde{\mathcal{U}}_\alpha = \left[\frac{1.45\alpha + 2.675}{-\alpha + 8.5}\ ;\ \frac{-1.45\alpha + 5.575}{\alpha + 6.5}\right]. \tag{65}$$

**(ii) Calculation of $\tilde{d}_\alpha$.**

Eq. (61) in (49), we obtain:

$$\tilde{d}_\alpha = [1.45\alpha + 2.675\ ;\ -1.45\alpha + 5.575]. \tag{66}$$

**(iii) Calculation of $\left(\widetilde{N}_S\right)_\alpha$**

Eq. (56) in (50), we obtain:

$$\left(\widetilde{N}_S\right)_\alpha = \left[\frac{\alpha+5}{-\alpha+8.5}\ ;\ \frac{-\alpha+7}{\alpha+6.5}\right]. \tag{67}$$

**(iv) Calculation of $\left(\widetilde{N}_f\right)_\alpha$**

Eq. (55) and (64) in (51); we obtain:

$$\left(\widetilde{N}_f\right)_\alpha = \frac{[2.45\alpha-0.575\ ;-2.45\alpha-4.325]}{[\alpha+6.5;\ -\alpha+8.5]} = [min\, S(\alpha);\ \ max\, S(\alpha)],$$

with $S(\alpha) = \{s_1(\alpha);\ s_2(\alpha);\ s_3(\alpha);\ s_4(\alpha)\}$; such that $s_1(\alpha) = \frac{2.45\alpha-0.575}{\alpha+6.5}$, $s_2(\alpha) = \frac{2.45\alpha-0.575}{-\alpha+8.5}$, $s_3(\alpha) = \frac{-2.45\alpha-4.325}{\alpha+6.5}$, and $s_4(\alpha) = \frac{-2.45\alpha-4.325}{-\alpha+8.5}$.

Solving these two PNLPs, gives: $min\, S(\alpha) = s_1(\alpha) = \frac{2.45\alpha-0.575}{\alpha+6.5}$ and $max\, S(\alpha) = s_3(\alpha) = \frac{-2.45\alpha-4.325}{\alpha+6.5}$. Hence,

$$\left(\widetilde{N}_f\right)_\alpha = \left[\frac{2.45\alpha-0.575}{\alpha+6.5}\ ;\ \frac{-2.45\alpha-4.325}{\alpha+6.5}\right]. \tag{68}$$

**(v) Calculation of $(\tilde{t}_S)_\alpha$**

Eq. (54) and (63) in (52), we obtain:

$$(\tilde{t}_S)_\alpha = \frac{[\alpha+5\ ;\ -\alpha+7]}{[0.55\alpha^2+20.65\alpha+9.7375\ ;\ 0.55\alpha^2-22.85\alpha+53.2375]}$$
$$= [min\, V(\alpha);\ \ max\, V(\alpha)],$$

with $V(\alpha) = \{v_1(\alpha);\ v_2(\alpha);\ v_3(\alpha);\ v_4(\alpha)\}$; such that $v_1(\alpha) = \frac{\alpha+5}{0.55\alpha^2+20.65\alpha+9.7375}$,

$v_2(\alpha) = \frac{\alpha+5}{0.55\alpha^2-22.85\alpha+53.2375}$, $v_3(\alpha) = \frac{-\alpha+7}{0.55\alpha^2+20.65\alpha+9.7375}$, and $v_4(\alpha) = \frac{-\alpha+7}{0.55\alpha^2-22.85\alpha+53.2375}$.

Solving these two PNLPs , gives :

$$\min V(\alpha) = v_2(\alpha) = \frac{\alpha+5}{0.55\alpha^2-22.85\alpha+53.2375},$$
$$max\, V(\alpha) = v_3(\alpha) = \frac{-\alpha+7}{0.55\alpha^2+20.65\alpha+9.7375}.$$

Hence

$$(\tilde{t}_S)_\alpha = \left[\frac{\alpha+5}{0.55\alpha^2-22.85\alpha+53.2375}\ ;\ \frac{-\alpha+7}{0.55\alpha^2+20.65\alpha+9.7375}\right]. \tag{69}$$

**(vi) Calculation of $\left(\tilde{t}_f\right)_\alpha$**

Eq. (63) and (64) in (53), we obtain :

$$\left(\tilde{t}_f\right)_\alpha = \frac{[2.45\alpha-0.575\ ;\ -2.45\alpha-4.325]}{[0.55\alpha^2+20.65\alpha+9.7375\ ;\ 0.55\alpha^2-22.85\alpha+53.2375]}$$

$$= [min\, Y(\alpha);\ \ max\, Y(\alpha)],$$

with $Y(\alpha) = \{y_1(\alpha);\ y_2(\alpha);\ y_3(\alpha);\ y_4(\alpha)\}$ such that

$$y_1(\alpha) = \frac{2.45\alpha - 0.575}{0.55\alpha^2 + 20.65\alpha + 9.7375}\ ; y_2(\alpha) = \frac{2.45\alpha - 0.575}{0.55\alpha^2 - 22.85\alpha + 53.2375},$$

$$y_3(\alpha) = \frac{-2.45\alpha - 4.325}{0.55\alpha^2 + 20.65\alpha + 9.7375} \text{ and } y_4(\alpha) = \frac{-2.45\alpha - 4.325}{0.55\alpha^2 - 22.85\alpha + 53.2375}.$$

Solving these two PNLPs , we obtain:

$$\min Y(\alpha) = y_1(\alpha) = \frac{2.45\alpha - 0.575}{0.55\alpha^2 + 20.65\alpha + 9.7375},$$
$$max\, Y(\alpha) = y_3(\alpha) = \frac{-2.45\alpha - 4.325}{0.55\alpha^2 + 20.65\alpha + 9.7375}.$$

Hence $$\left(\tilde{t}_f\right)_\alpha = \left[\frac{2.45\alpha - 0.575}{0.55\alpha^2 + 20.65\alpha + 9.7375}\ ;\ \frac{-2.45\alpha - 4.325}{0.55\alpha^2 + 20.65\alpha + 9.7375}\right] \quad (70)$$

Let's determine the supports and kernels (or modes) of $\tilde{\mathcal{U}};\ \tilde{d};\ \tilde{N}_S;\ \tilde{N}_f;\ \tilde{t}_S$ and $\tilde{t}_f$. The supports correspond to zero-level cuts and the cores (or modes) to level-one cuts. Thus, the zero level cuts ($i.e\ \ \alpha = 0$) in (65) ; (66) ; (67) ; (68) ; (69) and (70) are:

- $\tilde{\mathcal{U}}_0 = Supp(\tilde{\mathcal{U}}) = \left]\frac{2.675}{8.5}\ ;\ \frac{5.575}{6.5}\right[ = ]0.315\ ; 0.858[$
- $\tilde{d}_0 = Supp(\tilde{d}) = ]2.675\ ; 5.575[$
- $\left(\tilde{N}_S\right)_0 = Supp(\tilde{N}_S) = \left]\frac{5}{8.5}\ ;\ \frac{7}{6.5}\right[ = ]0.588\ ; 1.076[$
- $\left(\tilde{N}_f\right)_0 = Supp(\tilde{N}_f) = \left]\frac{-0.575}{6.5}\ ;\ \frac{4.325}{6.5}\right[ = ]-0.088\ ; 0.665[$
- $(\tilde{t}_S)_0 = Supp(\tilde{t}_S) = \left]\frac{5}{53.2375}\ ;\ \frac{7}{9.7375}\right[ = ]0.093\ ; 0.719[$
- $\left(\tilde{t}_f\right)_0 = Supp(\tilde{t}_f) = \left]\frac{-0.575}{9.7375}\ ;\ \frac{4.325}{9.7375}\right[ = ]-0.059\ ; 0.444[$

The level one cuts (i.e. $\alpha = 1$) in (65); (66); (67); (68); (69) and (70) are:

- $\tilde{\mathcal{U}}_1 = Noy(\tilde{\mathcal{U}}) = \frac{4.125}{7.5} = 0.55$
- $\left(\tilde{N}_f\right)_1 = Noy(\tilde{N}_f) = \frac{1.875}{7.5} = 0.25,$
- $\tilde{d}_1 = Noy(\tilde{d}) = 4.125$
- $(\tilde{t}_S)_1 = Noy(\tilde{t}_S) = \frac{6}{30.9375} = 0.193,$
- $\left(\tilde{N}_S\right)_1 = Noy(\tilde{N}_S) = \frac{6}{7.5} = 0.80$
- $\left(\tilde{t}_f\right)_1 = Noy(\tilde{t}_f) = \frac{1.875}{30.9375} = 0.06.$

We point out that these modal values correspond exactly to the average values of these indicators in the case of an ordinary Markovian M/M/1 waiting system with a priori impatience. Table 1 contains the bounds of the alpha-cuts of fuzzy numbers $\tilde{\mathcal{U}};\ \tilde{d};\ \tilde{N}_S;\ \tilde{N}_f;\ \tilde{t}_S$ and $\tilde{t}_f$.

**Table 1**. Bounds of the alpha-cuts of the performance indicators of the FM/FM/1 system for $0 \le \alpha \le 1$

| $\alpha$ | $\tilde{U}^L_\alpha$ | $\tilde{U}^U_\alpha$ | $\tilde{d}^L_\alpha$ | $\tilde{d}^U_\alpha$ | $(\tilde{N}_S)^L_\alpha$ | $(\tilde{N}_S)^U_\alpha$ | $(\tilde{N}_f)^L_\alpha$ | $(\tilde{N}_f)^U_\alpha$ | $(\tilde{t}_S)^L_\alpha$ | $(\tilde{t}_S)^U_\alpha$ | $(\tilde{t}_f)^L_\alpha$ | $(\tilde{t}_f)^U_\alpha$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0.315 | 0.858 | 2.675 | 5.575 | 0.588 | 1.076 | -0.088 | 0.665 | 0.094 | 0.719 | -0.059 | 0.444 |
| 0.1 | 0.336 | 0.823 | 2.820 | 5.430 | 0.607 | 1.045 | -0.050 | 0.618 | 0.100 | 0.584 | -0.028 | 0.346 |
| 0.2 | 0.357 | 0.789 | 2.965 | 5.285 | 0.627 | 1.015 | -0.012 | 0.572 | 0.108 | 0.490 | -0.006 | 0.276 |
| 0.3 | 0.379 | 0.756 | 3.110 | 5.140 | 0.646 | 0.985 | 0.024 | 0.528 | 0.114 | 0.419 | 0.010 | 0.225 |
| 0.4 | 0.402 | 0.724 | 3.255 | 4.995 | 0.666 | 0.957 | 0.059 | 0.485 | 0.122 | 0.365 | 0.022 | 0.185 |
| 0.5 | 0.425 | 0.693 | 3.400 | 4.850 | 0.688 | 0.929 | 0.093 | 0.443 | 0.131 | 0.323 | 0.032 | 0.153 |
| 0.6 | 0.449 | 0.663 | 3.705 | 4.705 | 0.709 | 0.901 | 0.126 | 0.402 | 0.141 | 0.287 | 0.040 | 0.128 |
| 0.7 | 0.473 | 0.633 | 3.690 | 4.560 | 0.731 | 0.875 | 0.158 | 0.363 | 0.152 | 0.258 | 0.047 | 0.106 |
| 0.8 | 0.498 | 0.605 | 3.835 | 4.415 | 0.753 | 0.849 | 0.190 | 0.324 | 0.164 | 0.233 | 0.052 | 0.089 |
| 0.9 | 0.524 | 0.577 | 3.980 | 4.270 | 0.776 | 0.824 | 0.220 | 0.286 | 0.178 | 0.212 | 0.057 | 0.074 |
| 1.0 | 0.550 | 0.550 | 4.125 | 4.125 | 0.800 | 0.800 | 0.25 | 0.25 | 0.193 | 0.193 | 0.060 | 0.060 |

Moreover, we can do without this table, by determining the membership function of each indicator by using relations (27) and (28). Thus

**1) Membership function of $\tilde{U}$.**

With regard to relation (65), we can write:

$$\mu_{\tilde{U}}(x) = \begin{cases} r_2^{-1}(x) & if \quad r_2(0) \le x \le r_2(1), \\ r_3^{-1}(x) & if \quad r_3(1) < x \le r_3(0), \\ 0 & otherwise. \end{cases}$$

We know that $r_2(\alpha) = \frac{1.45\alpha+2.675}{-\alpha+8.5}$ and $r_3(\alpha) = \frac{-1.45\alpha+5.575}{\alpha+6.5}$.

To find $r_2^{-1}(x)$ and $r_3^{-1}(x)$, we need to solve the following equations: $\frac{1.45\alpha+2.675}{-\alpha+8.5} = x$ and $\frac{-1.45\alpha+5.575}{\alpha+6.5} = x$. Solving these two equations gives: $r_2^{-1}(x) = \frac{8.5x+2.675}{x+1.45}$ and $r_3^{-1}(x) = \frac{5.575+6.5x}{1.45+x}$. From where

$$\mu_{\tilde{U}}(x) = \begin{cases} \frac{8.5x+2.675}{x+1.45} & if \quad 0.315 \le x \le 0.55, \\ \frac{5.575+6.5x}{1.45+x} & if \quad 0.55 < x \le 0.858, \\ 0 & otherwise. \end{cases}$$ )

The graph of $\mu_{\tilde{U}}(x)$ is given by figure 1. Figure 1 shows that $supp(\tilde{U})$ is approximately between 0.315 and 0.858. This means that the fuzzy rate of server utilization per hour is between 0.315 and 0.858 or between 31.5% and 85.8%. The fuzzy rate cannot be below 0.315 or above 0.858. The most frequent value is 0.55 per hour or 55%.

**2) Membership function of $\tilde{d}$.**

Starting from the relation (66), we can write:

$$\mu_{\tilde{d}}(x) = \begin{cases} h_1^{-1}(x) & if \quad h_1(0) \le x \le h_1(1) \\ h_2^{-1}(x) & if \quad h_2(1) < x \le h_2(0) \cdot \\ 0 & otherwise \end{cases}$$

With $h_1(\alpha) = 1.45\alpha + 2.675$ and $h_2(\alpha) = -1.45\alpha + 5.575$. To find $h_1^{-1}(x)$ and $h_2^{-1}(x)$, let

$1.45\alpha + 2.675 = x$ and $-1.45\alpha + 5.575 = x$. The resolution of these two equations, gives us: $h_1^{-1}(x) = \frac{x-2.267}{1.45}$ and $h_2^{-1}(x) = \frac{5.575-x}{1.45}$. Hence

$$\mu_{\tilde{d}}(x) = \begin{cases} \frac{x-2.675}{1.45} & if \quad 2.675 \leq x \leq 4.125, \\ \frac{5.575-x}{1.45} & if \quad 4.125 < x \leq 5.575, \\ 0 & otherwise. \end{cases} \quad (72)$$

The graph of $\mu_{\tilde{d}}(x)$ is given in Figure 2. The fuzzy throughput of the system is approximately between 2.675 and 5.575. This means that the fuzzy throughput of the system cannot be below 2 clients or exceed 6 clients per hour for the system to function properly. The most probable value is 4.125 or 4 customers per hour.

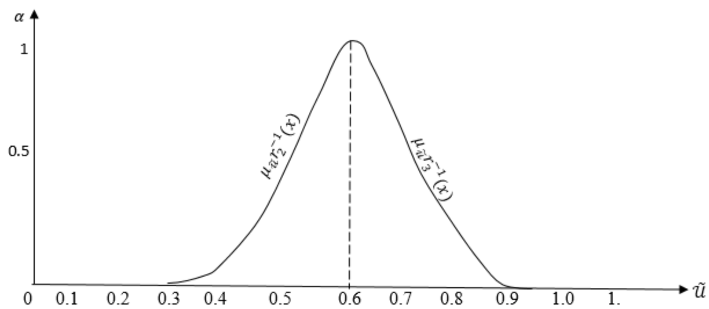


**Figure 1.** Membership function of the fuzzy server utilization rate

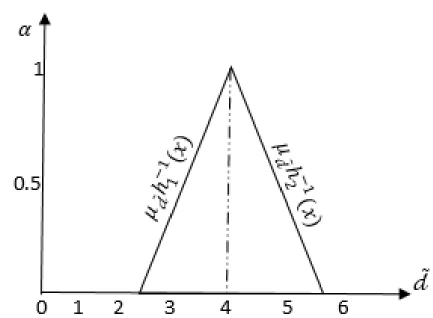


**Figure 2**. Membership function of the fuzzy flow of the system

**3) Membership function of $\tilde{N}_S$**

From relation (67), we can write:

$$\mu_{\tilde{N}_S}(x) = \begin{cases} p_1^{-1}(x) & if \quad p_1(0) \leq x \leq p_1(1), \\ p_2^{-1}(x) & if \quad p_2(1) < x \leq p_2(0), \\ 0 & otherwise. \end{cases}$$

with $p_1(\alpha) = \frac{\alpha+5}{-\alpha+8.5}$ and $p_2(\alpha) = \frac{-\alpha+7}{\alpha+6.5}$. To find $p_1^{-1}(x)$ and $p_2^{-1}(x)$, let us set $\frac{\alpha+5}{-\alpha+8.5} = x$ and $\frac{-\alpha+7}{\alpha+6.5} = x$. Solving these two equations gives us $p_1^{-1}(x) = \frac{8.5x-5}{1+x}$ and $p_2^{-1}(x) = \frac{7-6.5x}{1+x}$. Hence

$$\mu_{\tilde{N}_S}(x) = \begin{cases} \frac{8.5x-5}{1+x} & if \quad 0.588 \leq x \leq 0.8, \\ \frac{7-6.5x}{1+x} & if \quad 0.8 < x \leq 1.077, \\ 0 & otherwise. \end{cases} \tag{73}$$

The graph of $\mu_{\tilde{N}_S}(x)$ is given by figure 3. The fuzzy average number of customers in the system is approximately between 0.588 and 1.077. This means that the average number of customers in the system cannot be less than one customer or exceed two customers. The most possible value is 0.8 or one customer.

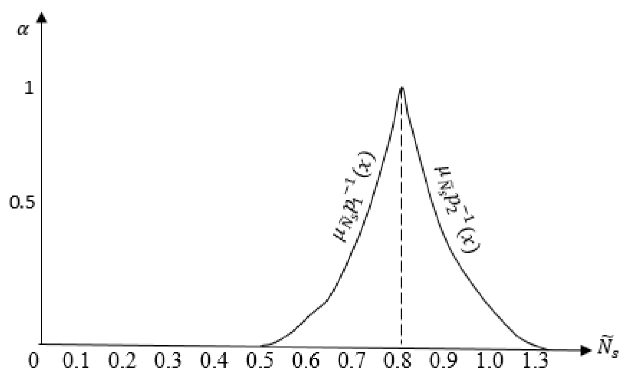


**Figure 3.** Membership function of the fuzzy average number of customers in the system

**4) Membership function of $\tilde{N}_f$**

Considering the relation (68), we can write:

$$\mu_{\tilde{N}_f}(x) = \begin{cases} s_1^{-1}(x) & if \quad s_1(0) \leq x \leq s_1(1), \\ s_3^{-1}(x) & if \quad s_3(1) < x \leq s_3(0), \\ 0 & otherwise, \end{cases}$$

with $s_1(\alpha) = \frac{1.45\alpha+2.675}{-\alpha+8.5}$ and $s_3(\alpha) = \frac{-1.45\alpha+5.575}{\alpha+6.5}$. Let us set $\frac{1.45\alpha+2.675}{-\alpha+8.5} = x$ and $\frac{-1.45\alpha+5.575}{\alpha+6.5} = x$. Solving these two equations gives us $s_1^{-1}(x) = \frac{8.5x-2.675}{1.45+x}$ and $s_3^{-1}(x) = \frac{5.575-6.5x}{1.45+x}$.

Hence
$$\mu_{\tilde{N}_f}(x) = \begin{cases} \frac{8.5x-2.675}{1.45+x} & if \ -0.088 \leq x \leq 0.25, \\ \frac{5.575-6.5x}{1.45+x} & if \quad 0.25 < x \leq 0.665, \\ 0 & otherwise. \end{cases} \tag{74}$$

The graph of $\mu_{\tilde{N}_f}(x)$ is given by figure 4. The fuzzy average number of customers in the queue is approximately between — 0.088 and 0.665, but the negative number of customers is meaningless (because $N_f \in \mathbb{N}$), so practically, this number is approximately only between 0 and 0.665. This is why, in figure 4, the corresponding area of negative values is hatched.

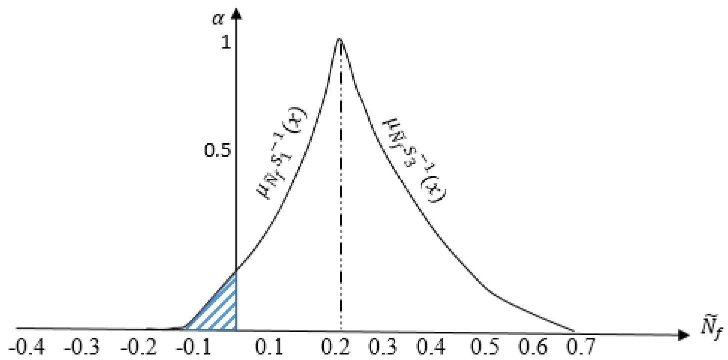


**Figure 4.** Membership function of the fuzzy average number of customers in the queue

**5) Membership function of $\tilde{t}_S$**

Starting from the relation (69), we can write:

$$\mu_{\tilde{t}_S}(x) = \begin{cases} v_2^{-1}(x) & if \quad v_2(0) \le x \le v_2(1), \\ v_3^{-1}(x) & if \quad v_3(1) < x \le v_3(0), \\ 0 & otherwise, \end{cases}$$

with $v_2(\alpha) = \frac{\alpha+5}{0.55\alpha^2-22.85\alpha+53.2375}$ and $v_3(\alpha) = \frac{-\alpha+7}{0.55\alpha^2+20.65\alpha+9.7375}$. Let us set: $\frac{-\alpha+7}{0.55\alpha^2+20.65\alpha+9.7375} = x$ and $\frac{\alpha+5}{0.55\alpha^2-22.85\alpha+53.2375} = x$. The development of these two equations brings us back to the resolution of two second-degree equations in $\alpha$:

$$0.55x\alpha^2 - (22.85x + 1)\alpha + 53.2375x - 5 = 0 \text{ and}$$
$$0.55x\alpha^2 + (20.65x + 1)\alpha + 9.7375x - 7 = 0$$

The solutions are as follows :

$$\alpha_{1,2} = \frac{(22.85x+1)\pm\sqrt{405x^2+56.7x+1}}{1.1x} \text{ for the first equation and}$$

$$\alpha_{1,2} = \frac{-(20.65x+1)\pm\sqrt{405x^2+56.7x+1}}{1.1x} \text{ for the second.}$$

In the case of the first equation, the function $\alpha_2 = \frac{22.85x+1-\sqrt{405x^2+56.7x+1}}{1.1x}$ is chosen as the membership function, because it provides values in the interval $[0\,;\,1]$ for $x \in [0.094\,;\,0.193]$, while for the second equation, for $x \in [0.193\,;\,0.719]$, the values in the interval $[0\,;\,1]$ are provided by the function $\alpha_1 = \frac{-20.65x-1+\sqrt{405x^2+56.7x+1}}{1.1x}$. We then have

$$v_2^{-1}(x) = \frac{22.85x+1-\sqrt{405x^2+56.7x+1}}{1.1x} \text{ and } v_3^{-1}(x) = \frac{-20.65x-1+\sqrt{405x^2+56.7x+1}}{1.1x}.$$

Finally,

$$\mu_{\tilde{t}_S}(x) = \begin{cases} \frac{22.85x+1-\sqrt{405x^2+56.7x+1}}{1.1x} & if \quad 0.094 \le x \le 0.193, \\ \frac{-20.65x-1+\sqrt{405x^2+56.7x+1}}{1.1x} & if \quad 0.193 < x \le 0.719, \\ 0 & otherwise. \end{cases} \tag{75}$$

The graph of $\mu_{\tilde{t}_S}(x)$ (x) is given by figure 5. The average fuzzy waiting time of customers in the system is approximately between 0.094 and 0.719 hours, or between 5 and 43 minutes. The most likely value is 0.193 hours, or 11 minutes.

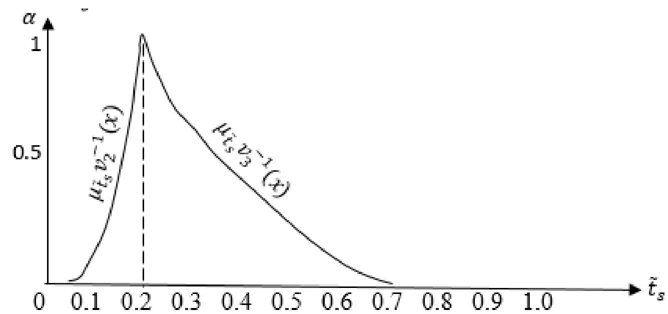


**Figure 5**. Membership function of the average fuzzy waiting time of customers in the system

**f) Membership function of $\tilde{t}_f$**

Based on the relation (70), we can write:

$$\mu_{\tilde{t}_f}(x) = \begin{cases} y_1^{-1}(x) & if \quad y_1(0) \le x \le y_1(1), \\ y_3^{-1}(x) & if \quad y_3(1) < x \le y_3(0), \\ 0 & otherwise, \end{cases}$$

with $y_1(\alpha) = \frac{2.45\alpha-0.575}{0.55\alpha^2+20.65\alpha+9.7375}$ and $y_3(\alpha) = \frac{-2.45\alpha+4.325}{0.55\alpha^2+20.65\alpha+9.7375}$. Let us set $\frac{2.45\alpha-0.575}{0.55\alpha^2+20.65\alpha+9.7375} = x$ and $\frac{-2.45\alpha+4.325}{0.55\alpha^2+20.65\alpha+9.7375} = x$. By expanding these two equations, we find two quadratic equations in α:

$$0.55x\alpha^2 + (20.65x - 2.45)\alpha + 9.7375x + 0.575 = 0 \text{ and}$$
$$0.55x\alpha^2 + (20.65x + 2.45)\alpha + 9.7375x - 4.325 = 0.$$

The solutions are as follows:

$$\alpha_{1,2} = \frac{-(20.65x-2.45)\pm\sqrt{405x^2+102.45x+6.0025}}{1.1x} \text{ for the first equation and}$$

$$\alpha_{1,2} = \frac{-(20.65x+2.45)\pm\sqrt{405x^2+110.7x+6.0025}}{1.1x} \text{ for the second.}$$

For the first equation, the function $\alpha_2 = \frac{-(20.65x-2.45)-\sqrt{405x^2+102.45x+6.0025}}{1.1x}$ is chosen as the membership function, because it provides values in the interval $[0\,;\,1]$ for $x \in [-0.059\,;\,0.060]$,

while for $x \in [0.060\,;\,0.444]$, for the second equation, values in the interval $[0\,;\,1]$ are provided
by the function $\alpha_1 = \frac{-(20.65x+2.45)+\sqrt{405x^2+110.7x+6.0025}}{1.1x}$. Then we have,

$$y_1^{-1}(x) = \frac{-(20.65x-2.45)-\sqrt{405x^2+102.45x+6.0025}}{1.1x} \text{ and } y_3^{-1}(x) = \frac{-(20.65x+2.45)+\sqrt{405x^2+110.7x+6.0025}}{1.1x}.$$

Hence

$$\mu_{\tilde{t}_f}(x) = \begin{cases} \frac{-(20.65x-2.45)-\sqrt{405x^2+102.45x+6.0025}}{1.1x} & if\ -0.059 \leq x \leq 0.060, \\ \frac{-(20.65x+2.45)+\sqrt{405x^2+110.7x+6.0025}}{1.1x} & if\ 0.060 < x \leq 0.444, \\ 0 & otherwise. \end{cases} \quad (76)$$

The graph of $\mu_{\tilde{t}_f}(x)$ is given in Figure 6. The average fuzzy waiting time of customers in the queue is approximately between — 0.059 and 0.444 hours, but for the same reason mentioned above, time is a variable always greater than or equal to zero, (i.e. $t \in \mathbb{R}^+$), practically, the average fuzzy waiting time in the queue is approximately between 0 and 0.444 hours, or between 0 and 26 minutes. Thus, in Figure 6, the corresponding area of negative values is hatched. The most favorable value is 0.06 hours, or 4 minutes.

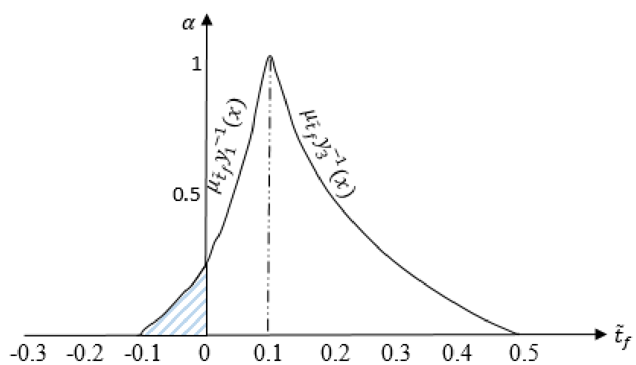


**Figure 6.** Membership function of the average fuzzy waiting time in the queue

## 4. DISCUSSIONS

In steady state, starting from the numerical application above, the soft alpha-cut method provides the fuzzy Markov queueing system FM/FM/1 with a priori impatience with some variants as follows: the performance indicators of this fuzzy Markov queueing system are fuzzy numbers and not average values as for the ordinary Markov queueing system M/M/1 with a priori impatience. The alpha-cuts of the indicators are closed intervals of $\mathbb{R}$ where the limits are continuous functions $L$ and $U$ in $\alpha$ respectively increasing and decreasing.

The supports of the indicators of this fuzzy Markov queueing system are open intervals whose limits are real numbers. In this case, managers have a wide range of maneuvers for their decision-making, in other words, the decision of managers of this type of systems is flexible. On the other hand, for the ordinary Markovian queueing system M/M/1 with a priori impatience, the decision-making is restricted to a fixed position, i.e. is not the subject of any debate.

The modes of the indicators of this fuzzy Markovian queueing system are real numbers that correspond exactly to the mean values of the ordinary Markovian queueing system M/M/1 with a priori impatience. The membership functions of the indicators of this fuzzy Markovian queueing system are continuous reciprocal functions $L^{-1}$ and $U^{-1}$ respectively increasing and decreasing. The graphical representations of the membership functions of the indicators of this fuzzy Markovian queueing system are made in the plane where the abscissa axis is carried by the indicator in question and the ordinate axis by the different values of α with $\alpha \in [0; 1]$.

## 5. CONCLUSION

This article set itself the objective of analyzing the performance of the Markovian queueing system FM/FM/1 with a priori impatience, in steady state, by the approach of alpha - cuts and intervals. To achieve this, we recalled some preliminaries of the theory of fuzzy sets capable of facilitating the study of this system in order to evaluate its performance indicators. The main tool on which we relied is the method of soft alpha - cuts. This is a method based on the arithmetic of alpha - cuts and intervals. A numerical application was proposed to illustrate the validity and practicability of this method.

From this numerical application, it emerged that all the values of the performance indicators similar to those calculated in the classical model were obtained. Each indicator is characterized by its support, its mode and by its membership function which made possible the graphical representation of each indicator. The fuzzy state of these performance indicators shows that a Markovian queueing system with impatient customers in which imprecise information is introduced keeps this imprecision throughout its process. Compared to the results obtained in a classical model to those obtained the fuzzy models, the average values of the classical models correspond exactly to the modal values of the fuzzy models, therefore, the classical models are sub – models of the fuzzy models.

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