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Local Metric Dimension of Certain Operation of Generalized Petersen Graph

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Abstract

A subset W of $V(G)$ is called a local resolving set of G if $r(u|W) \neq r(v|W)$ for every two adjacent vertices $u, v \in V(G)$. The smallest cardinality of all local resolving sets in G is called the local metric dimension of G, denoted by $\text{lmd}(G)$. The local resolving set of G with cardinality $\text{lmd}(G)$ is called a local basis of G. In this paper, we present a novel study, a topic that has not been extensively explored in previous research, on the local metric dimension of certain operation of generalized Petersen graph $sP_{n,1}$ and determine the lower and upper bounds of $\text{lmd}(sP_{n,m})$ with $n \geq 3$, $s \geq 1$, and $1 \leq m \leq \left\lfloor \frac{n-1}{2} \right\rfloor$. We also show that the lower bound is sharp.

Keywords: local resolving set; local metric dimension; generalized Petersen graph.

Abstrak

Suatu subset W dari $V(G)$ dikatakan himpunan pembeda lokal dari G jika $r(u|W) \neq r(v|W)$ untuk setiap dua titik bertetangga $u, v \in V(G)$. Kardinalitas terkecil dari semua himpunan pembeda lokal di G disebut dimensi metrik lokal dari G, dinotasikan $\text{lmd}(G)$. Himpunan pembeda lokal G dengan kardinalitas $\text{lmd}(G)$ disebut basis lokal dari G. Pada artikel ini, disajikan sebuah studi baru, topik yang belum dieskplorasi secara ekstensif dalam penelitian sebelumnya, tentang dimensi metrik lokal dari graf hasil operasi tertentu untuk graf Petersen diperumum $sP_{n,1}$ dan menentukan batas atas dan bawah dari $\text{lmd}(sP_{n,m})$ dengan $n \geq 3$, $s \geq 1$, dan $1 \leq m \leq \left\lfloor \frac{n-1}{2} \right\rfloor$. Kami juga menunjukkan bahwa batas bawah tersebut tajam.

Kata Kunci: *himpunan pembeda local; dimensi metrik local; graf Petersen diperumum.*

2020MSC: 05C12, 05C76

1. INTRODUCTION

All graphs considered in this paper are finite, simple, and connected. Let G be a graph. The vertex set of G is denoted by $V(G)$, and the edge set is denoted by $E(G)$. Two adjacent vertices u and v in G are written as $u \sim_G v$. If u and v are not adjacent in G, we write $u \nsim_G v$. The distance between two vertices u and v in G is the length of the shortest path in G that connects u and v, denoted by $d(u, v)$. Let $W = \{w_1, w_2, \dots w_k\}$ be a subset of $V(G)$. The representation of vertex $v \in V(G)$ with respect to W, denoted by $r(v|W)$, is defined as k-vector $(d(v, w_1), d(v, w_2), \dots, d(v, w_k))$. So, $r(v|W) = (d(v, w_1), d(v, w_2), \dots, d(v, w_k))$. A set W is called a resolving set of G if every two distinct vertices $u, v \in V(G)$ satisfies $r(u|W) \neq r(v|W)$. The smallest cardinality of all resolving sets in G is called the metric dimension of G, denoted by $\dim(G)$. The resolving set of G with cardinality $\dim(G)$ is called a basis of G.

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The topic of metric dimension in graph theory was first introduced by Harary and Melter [1], and separately by Slater [2]. The concept of metric dimension can be applied in several fields, such as problems related to chemical structures [3], or robot navigation problems [4]. Several studies that examine the metric dimension of a particular graph can be seen in [5][6][7][8]. One interesting thing for several authors related to this topic is studying the metric dimension of a graph obtained from graph operations [9][10][11][12]. The relationship between the metric dimension of a graph obtained from graph operations and its origin graph is shown in that study.

This article's main study is a graph's local metric dimension. This study is another version of a metric dimension problem. A set W of $V(G)$ is called a local resolving set of G if $r(u|W) \neq r(v|W)$ for every two adjacent vertices $u, v \in V(G)$. The smallest cardinality of all local resolving sets in G is called the local metric dimension of G, denoted by $\text{lmd}(G)$. The local resolving set of G with cardinality $\text{lmd}(G)$ is called a local basis of G. Okamoto et al. first studied the concept of local metric dimension [13]. One of the results stated by Okamoto et al. are as follows.

Theorem 1. [13] Let G be a nontrivial connected graph of order n. $\text{lmd}\ (G) = 1$ if and only if G is a bipartite graph.

Several authors have determined the local metric dimension of a particular graph [14][15][16]. A study of the relationship between the local metric dimension of a graph obtained from graph operations and its origin graph can be seen here [17][18][19][20].

The generalized Petersen graph was introduced by Watkins [21]. The generalized Petersen graph, denoted by $P_{n,m}$ with $n \geq 3$ and $1 \leq m \leq \left\lfloor \frac{n-1}{2} \right\rfloor$, is a graph with the vertex set $V(P_{n,m}) = \{u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_n\}$ and all edges of $P_{n,m}$ are in the form $u_i u_{i+1}, u_i v_i$, and $v_i v_{i+m}$ with indices taken modulo n. An example of a generalized Petersen graph is represented in Figure 1.

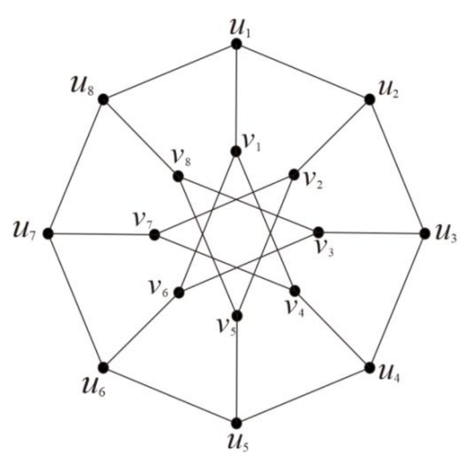

Figure 1. Generalized Petersen graph $P_{8,3}$

Some results regarding the metric dimension as well as the local metric dimension of the generalized Petersen graph have been obtained [22][23][24].

Asmiati et al. [25] defined a certain operation for the Petersen graph $P_{n,m}$ and then determined its metric dimension for $m = 1$. In this paper, we consider a novel study, a topic that has not been extensively explored in previous research, on the local metric dimension. We determine the lower and upper bounds of the local metric dimension of certain operation of generalized Petersen graph $sP_{n,m}$ and show that the lower bound is sharp.

2. DEFINITIONS

Definition 1. For $s \geq 1, n \geq 3$, and $1 \leq m \leq \left\lfloor \frac{n-1}{2} \right\rfloor$, a graph of certain operation for a generalized Petersen graphs $P_{n,m}$, denoted by $sP_{n,m}$, is a graph consisting of s Petersen graphs $P_{n,m}$ with the vertex set $V(sP_{n,m}) = \{u_i^j \mid 1 \leq i \leq n \text{ and } 1 \leq j \leq s\} \cup \{v_i^j \mid 1 \leq i \leq n \text{ and } 1 \leq j \leq s\}$ and the edge set contains all edges of the form $u_i^j u_{i+1}^j, u_i^j v_i^j, v_i^s v_{i+m}^s$, and $u_i^j u_i^{j+1}$ with index i taken from modulo n and $1 \leq j \leq s$.

For $sP_{n,m}$, we write $P_{n,m}^j$, $1 \leq j \leq s$, to represent the j-th Petersen graph $P_{n,m}$ in $sP_{n,m}$. We denote the vertex set in $P_{n,m}^j$ by $V(P_{n,m}^j)$. A vertex $u_i^j \subset V(sP_{n,m})$ represents a vertex $u_i, 1 \leq i \leq n$, in $P_{n,m}^j$. A vertex $v_i^j \subset V(sP_{n,m})$ represents a vertex $v_i, 1 \leq i \leq n$, in $P_{n,m}^j$. The following is a graph of certain operations for a generalized Petersen graph.

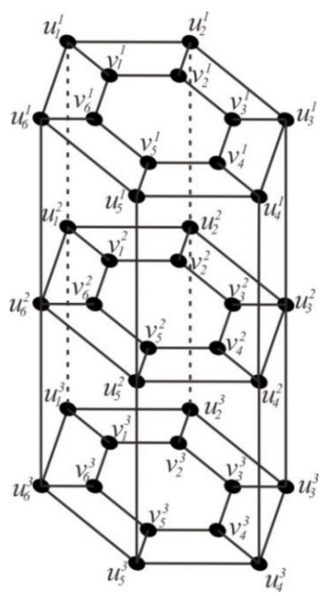

Figure 2. Graph of certain operation of generalized Petersen graph $3P_{6,1}$

3. RESULTS

In this section, we determine the local metric dimension of certain operation of generalized Petersen graph $sP_{n,1}$. Then, we provide the lower and upper bounds of the local metric dimension of $sP_{n,m}$. First, we show the local metric dimension of $sP_{n,1}$.

Theorem 2. Let $sP_{n,1}$ be a graph of certain operation of generalized Petersen graph with $s \geq 1$ and $n \geq 3$, then

$$\text{lmd}(sP_{n,1}) = \begin{cases} 1, \text{ if } n \text{ is even} \\ 2, \text{ if } n \text{ is odd.} \end{cases}$$

Proof.

Case 1: n is even.

Let B_1 and B_2 be subsets of $V(sP_{n,1})$. All vertices of $V(sP_{n,1})$ that belong to B_1 are as follows:

1. v_{2i+1}^{2j+1} with $0 \le i \le \frac{n}{2} - 1$ and $0 \le j \le \frac{s-1}{2}$, if s is odd or $0 \le j \le \frac{s}{2} - 1$, if s is even.
2. v_{2i}^{2j} with $1 \le i \le \frac{n}{2}$ and $1 \le j \le \frac{s-1}{2}$, if s is odd, or $1 \le j \le \frac{s}{2}$, if s is even.
3. $u_{2i+1}^{2j}$with $0 \le i \le \frac{n}{2} - 1$ and $1 \le j \le \frac{s-1}{2}$, if s is odd, or $1 \le j \le \frac{s}{2}$, if s is even.
4. u_{2i}^{2j+1} with $1 \le i \le \frac{n}{2}$ and $0 \le j \le \frac{s-1}{2}$, if s is odd, or $0 \le j \le \frac{s}{2} - 1$, if s is even.

All vertices of $V(sP_{n,1})$ that belong to B_2 are as follows:

1. v_{2i+1}^{2j} with $0 \le i \le \frac{n}{2} - 1$ and $1 \le j \le \frac{s-1}{2}$, if s is odd or $1 \le j \le \frac{s}{2}$, if s is even.
2. v_{2i}^{2j+1} with $1 \le i \le \frac{n}{2}$ and $0 \le j \le \frac{s-1}{2}$, if s is odd or $0 \le j \le \frac{s}{2} - 1$, if s is even.
3. u_{2i+1}^{2j+1} with $0 \le i \le \frac{n}{2} - 1$ and $0 \le j \le \frac{s-1}{2}$, if s is odd or $0 \le j \le \frac{s}{2} - 1$, if s is even.
4. u_{2i}^{2j} with $1 \le i \le \frac{n}{2}$ and $1 \le j \le \frac{s-1}{2}$, if s is odd or $1 \le j \le \frac{s}{2}$, if s is even.

Note that for every two vertices x and y in B_1, then $x \nsim_{sP_{n,1}} y$. Likewise in B_2, for every two vertices x and y in B_2, then $x \nsim_{sP_{n,m}} y$. Also, note that each vertex in B_1 is adjacent to a vertex in B_2, and conversely, each vertex in B_2 is adjacent to a vertex in B_1. Thus, it is concluded that for even n, $sP_{n,1}$ is a bipartite graph. Based on Theorem 1, for even n, lmd $(sP_{n,1}) = 1$.

Case 2: n is odd.

Let $W = \{u_1^1, u_{k+1}^1\}$ with $k = \frac{n-1}{2}$. Note that for every $s \ge 2$, a vertex u_1^s is the closest vertex to u_1^1 compared to all vertices in the s-th $P_{n,1}$, and a vertex u_{k+1}^s is the closest vertex to u_{k+1}^1 compared to all vertices in the s-th $P_{n,1}$. Therefore, for every $s \ge 2$, then $r(u_1^s|W) \ne r(a|W)$ and $r(u_{k+1}^s|W) \ne r(a|W)$ with a is any vertex in s-th $P_{n,1}$. Also, note that the shortest path $u_1^1 - a$ of length $d(u_1^1, a)$, where a is any vertex in s-th $P_{n,1}$, must pass through a vertex u_1^s. Similarly, the shortest path $u_{k+1}^1 - a$ of length $d(u_{k+1}^1, a)$, where a is any vertex in s-th $P_{n,1}$, must pass through a vertex u_{k+1}^s. Therefore, the representation of adjacent vertices in $sP_{n,1}$ with respect to W is given by:

1. u_i^s and u_{i+1}^s

$$r(u_i^s|W) = \begin{cases} (i-1+(s-1), k-i+1+(s-1)), & \text{if } 2 \le i \le k \\ (2k-i+2+(s-1), i-(k+1)+(s-1)), & \text{if } k+2 \le i \le 2k+1 \end{cases}$$

and

$$r(u_{i+1}^s|W) = \begin{cases} (i+(s-1), k-i+(s-1)), & \text{if } 2 \le i \le k \\ (2k-i+1+(s-1), i-k+(s-1)), & \text{if } k+2 \le i \le 2k+1. \end{cases}$$

2. u_i^s and v_i^s

$$r(u_i^s|W) = \begin{cases} (i-1+(s-1), k-i+1+(s-1)), & \text{if } 2 \le i \le k \\ (2k-i+2+(s-1), i-(k+1)+(s-1)), & \text{if } k+2 \le i \le 2k+1 \end{cases}$$

and

$$r(v_i^s|W) = \begin{cases} \big(i+(s-1), k-i+2+(s-1)\big), & \text{if } 2 \le i \le k \\ (2k-i+3+(s-1), i-k+(s-1)), & \text{if } k+2 \le i \le 2k+1. \end{cases}$$

3. v_i^s and v_{i+1}^s

$$r(v_i^s|W) = \begin{cases} \big(i+(s-1), k-i+2+(s-1)\big), & \text{if } 1 \le i \le k+1 \\ (2k-i+3+(s-1), i-k+(s-1)), & \text{if } k+2 \le i \le 2k+1 \end{cases}$$

and

$$r(v_{i+1}^s|W) = \begin{cases} \big(i+1+(s-1), k-i+1+(s-1)\big), & \text{if } 1 \le i \le k+1 \\ (2k-i+2+(s-1), i-k+1+(s-1)), & \text{if } k+2 \le i \le 2k+1. \end{cases}$$

4. u_i^s and u_i^{s+1}

$$r(u_i^s|W) = \begin{cases} \big(i-1+(s-1), k-i+1+(s-1)\big), & \text{if } 1 \le i \le k+1 \\ (2k-i+2+(s-1), i-(k+1)+(s-1)), & \text{if } k+2 \le i \le 2k+1 \end{cases}$$

and

$$r\big(u_i^{s+1}|W\big) = \begin{cases} (i-1+s, k-i+1+s), & \text{if } 1 \le i \le k+1 \\ (2k-i+2+s, i-(k+1)+s), & \text{if } k+2 \le i \le 2k+1. \end{cases}$$

Based on the results above, it is concluded that W is a local resolving set of $sP_{n,1}$. Thus, that for odd n, lmd $(sP_{n,1}) \le 2$. Note that $sP_{n,1}$ is not a bipartite graph for odd n. By Theorem 1, we obtain that lmd $(sP_{n,1}) \ge 2$. So, for odd n, lmd $(sP_{n,1}) = 2$. ■

In Theorem 3, we show the lower bound of lmd$(sP_{n,m})$ which is related to parameter lmd$(P_{n,m})$.

Theorem 3. Let $sP_{n,m}$ be a graph of certain operation of generalized Petersen graph $P_{n,m}$ with $s \ge 1, n \ge 3,$ and $1 \le m \le \left\lfloor \frac{n-1}{2} \right\rfloor$. Then lmd$(sP_{n,m}) \ge$ lmd$(P_{n,m})$.

Proof.
Let W and W' be local bases of $P_{n,m}$ and $sP_{n,m}$, respectively. Suppose lmd$(sP_{n,m}) <$ lmd$(P_{n,m})$, then $|W'| < |W|$. Since W is a local basis of $P_{n,m}$, any subset of $V(P_{n,m})$ whose cardinality is smaller than $|W|$ is not a local resolving set of $P_{n,m}$. Therefore, in $sP_{n,m}$, there exist two adjacent verticess x and y in $P_{n,m}^j$ such that $r(x|W') = r(y|W')$ for every j with $1 \le j \le s$. Hence, W' is not a local resolving set of $sP_{n,m}$, a contradiction . ■

Next, we give a lemma stating the existence of a generalized Petersen graph $P_{n,m}$ with its local metric dimension equal to the local metric dimension of $sP_{n,m}$.

Lemma 4. There exists a generalized Petersen graph $P_{n,m}$ such that lmd$(sP_{n,m}) =$ lmd$(P_{n,m})$.

Proof.
We consider the generalized Petersen graph $P_{n,1}$. In Theorem 2, we obtain that for even n, lmd $(sP_{n,1}) = 1$ and for odd n, lmd $(sP_{n,1}) = 2$. Next, we determine the local metric dimension of $P_{n,1}$. In determining lmd $(P_{n,1})$, we consider two cases.

Case 1: n is odd

For $P_{n,1}$ with n is odd, note that $P_{n,1}$ is not a bipartite graph. By Theorem 1, we obtain that for odd n, lmd $\left(P_{n,1}\right) \geq 2$. Since lmd $\left(sP_{n,1}\right) = 2$ for odd n, by Theorem 3, we conclude that lmd $\left(P_{n,1}\right) \leq 2$ for odd n. Therefore, for odd n, lmd$(P_{n,1}) = 2$.

Case 2: n is even

We have lmd $\left(sP_{n,1}\right) = 1$ for even n. By Theorem 3, we conclude that for even n, lmd $\left(P_{n,1}\right) = 1$.

From the two cases above, we have $lmd\left(sP_{n,1}\right) = lmd\left(P_{n,1}\right)$. ∎

Lemma 4 above shows that the lower bound of $lmd(sP_{n,m})$ in Theorem 3 is sharp.

The following theorem shows the upper bound of lmd$(sP_{n,m})$. Before that, we define the vertex set of $P_{n,m}$ as $V\left(P_{n,m}\right) = U' \cup V'$, where $U' = \{u_1, u_2, \ldots, u_n\}$ and $V' = \{v_1, v_2, \ldots, v_n\}$. If W is a local basis of $P_{n,m}$, we write $W_{U'}$ to represent all vertices of U' that belong to W. Similarly, we write $W_{V'}$ to represent all vertices of V' that belong to W.

Theorem 5. Let $sP_{n,m}$ be a graph of certain operation of generalized Petersen graph with $s \geq 1$, $n \geq 3$ and $1 \leq m \leq \left\lfloor \frac{n-1}{2} \right\rfloor$. If W is a local basis of $P_{n,m}$ with $W = W_{U'} \cup W_{V'}$ and $|W_{V'}| = d$, then $lmd(sP_{n,m}) \leq lmd(P_{n,m}) + (s-1)d$.

Proof.

Let $W' \subset V(P_{n,m}^1)$ and $D \subset V(sP_{n,m})$. Next, we define $W' = \{u_i^1 \mid u_i \in W_{U'}\} \cup \{v_i^1 \mid v_i \in W_{V'}\}$ and $D = \{v_i^j \mid v_i \in W_{V'}, 2 \leq j \leq s\}$. Note that $|W'| = |W|$ and $|D| = (s-1)d$ with $d = |W_{V'}|$. Then, we show that $W' \cup D$ is a local resolving set of $sP_{n,m}$. Since W is a local basis of $P_{n,m}$, W' locally resolves all adjacent vertices in $P_{n,m}^1$. Note that for every j with $2 \leq j \leq s$, a vertex u_i^j is the only closest vertex to u_i^1 compared to all vertices in $P_{n,m}^j$. Thus, if W' is combined with the set of vertex corresponding to $W_{V'}$ in every $P_{n,m}^j$, which in this case are elements of the set D, then $W' \cup D$ locally resolves any two adjacent vertices in $P_{n,m}^j$, for every j with $2 \leq j \leq s$. In other words, if in each $P_{n,m}^j$, $2 \leq j \leq s$, there is an element of D, then for any two adjacent vertices $x, y \in V\left(sP_{n,m}\right)$, we obtain $r(x \mid W' \cup D) \neq r(y \mid W' \cup D)$. Thus, $W' \cup D$ is a local resolving set of $sP_{n,m}$. We conclude that lmd$(sP_{n,m}) \leq$ lmd$(P_{n,m}) + (s-1)d$. ∎

By combining Theorem 3 and Theorem 5, we have a corollary below.

Corollary 6. Let $sP_{n,m}$ be a graph of certain operation of generalized Petersen graph with $s \geq 1, n \geq 3$, and $1 \leq m \leq \left\lfloor \frac{n-1}{2} \right\rfloor$. If W is a local basis of $P_{n,m}$ with $W = W_{U'} \cup W_{V'}$ and $|W_{V'}| = d$, then lmd$(P_{n,m}) \leq$ lmd$(sP_{n,m}) \leq$ lmd$(P_{n,m}) + (s-1)d$.

4. DISCUSSION

The findings of this study contribute to a broader understanding of the local metric dimension in graph operations, especially in the case of generalized Petersen graph. One of the key insights in our study is that the graph $sP_{n,m}$ with $m = 1$ is a graph with constant local metric dimension. We show that $sP_{n,m}$ with even n is a bipartite graph. Previously, Okamoto [13] has studied the local metric dimension of bipartite graph. This study helped us to show that the local metric dimension of $sP_{n,m}$ is 1 for even n. As for odd n, the graph $sP_{n,m}$ is not a bipartite graph. Furthermore, we obtain that the local metric dimension of $sP_{n,m}$ is 2 for odd n.

In addition, our study provides the upper and lower bounds of the local metric dimension of this graph operation. The sharpness of the lower bound is shown by the existence of a generalized Petersen graph $P_{n,m}$ with lmd$(P_{n,m}) = \text{lmd}(sP_{n,m})$. This is because at the given lower bound, the parameter used is the local metric dimension of $P_{n,m}$. The use of this parameter is in line with previous studies, including studies conducted by Saputro et al. (2012, 2013, 2017), which explored the metric dimension in various graph operations [9][10][11]. At the upper bound we provide, in addition to parameter lmd($P_{n,m}$), there are other parameters related to the local basis of $P_{n,m}$. Furthermore, this study lays the groundwork for investigating whether there are sharper upper bound for the local metric dimension of certain operation of generalized Petersen graph, which reduce the dependence on some parameters.

Future studies can explore whether the same bounds apply to different variations of Petersen graph, including those with alternative edge modifications. Overall, these findings contribute to ongoing research in discrete mathematics and graph theory. In particular, they offer potential applications in network optimization, secure communication networks, and robotic navigation systems.

5. CONCLUSION

In this paper, we determine the local metric dimension of certain operations of generalized Petersen graph, especially for $P_{n,1}$. We obtain that for even n, lmd $(sP_{n,1}) = 1$ and for odd n, lmd $(sP_{n,1}) = 2$. We also determine the general bounds of the local metric dimension of $sP_{n,m}$. We conclude that lmd$(P_{n,m}) \leq$ lmd$(sP_{n,m}) \leq$ lmd$(P_{n,m}) + (s-1)d$ with $d = |W_{V'}|$.

For the lower bound, which is related only to parameter lmd$(P_{n,m})$, we also show the existence of generalized Petersen graph $P_{n,m}$ with its local metric dimension equal to the local metric dimension of $sP_{n,m}$. It means that the lower bound is sharp. For the upper bound, we give a bound that relates not only to parameter lmd$(P_{n,m})$ but also to parameter $|W_{V'}|$. So, we provide two parameters on the upper bound. A question for future study is whether there is a better upper bound than the one we offered, containing only one parameter. It would be interesting to find an upper bound of $sP_{n,m}$ containing only one parameter, and show that the upper bound is sharp.

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