The Structures of Non-Coprime Graphs for Finite Groups from Dihedral Groups with Regular Composite Orders

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Abstract
For any finite group, the non-coprime graph of the group is a graph with vertices consisting of all non-identity elements of the group. Two different vertices are considered adjacent if their orders are not coprime, meaning their greatest common divisor (gcd) is not equal to one. Misuki provides the structure of the non-coprime graph for the dihedral group when the order is a prime power. We establish a more general property for cases where the order of the group is a regular composite number, discovering that the structure of the non-coprime graph for a dihedral group can be partitioned into some number of complete subgraphs.

Keywords: dihedral group; disjoint subgraph; non-coprime graph.

1. INTRODUCTION

The current focus of research revolves around the graph representation of finite groups. Ma conducted a study related to group representation called coprime graphs [1], then Mansoori 2016 also conducted research related to group representation called non-coprime graphs [2], then new studies emerged related to more specific group representations in graphs. For example, studies have already been conducted on the characterizations and numerical invariants of several graph representations, including their topological indices. These studies include the coprime graph from a dihedral group, an integer group modulo, or a generalized quaternion group, as well as non-coprime graphs from a dihedral group [3]–[10].

In addition to the mentioned studies, research has also explored various other types of graphs, such as prime graphs, intersection graphs, nilpotent graphs, and commuting graphs. These...
investigations have sought to understand the properties, characteristics, and numerical invariants associated with these graph structures. For instance, prime graphs examine the relationships between elements in a group based on their prime divisors, while intersection graphs explore the intersections of subgroups within a given group. Commuting graphs, on the other hand, investigate the relationships between group elements that commute with each other. These diverse graph representations provide valuable insights into different aspects of group theory and its applications in mathematics [11]–[19].

2. METHODS

This research begins by conducting a literature study, namely finding and studying reading sources related to the topic. After conducting a literature study, non-coprime graphs were shaped from dihedral groups for several cases. Then from the case that is constructed, we looked for a pattern as a conjecture, and the conjectures are stated as a theorem after we prove it.

3. RESULTS AND DISCUSSIONS

The dihedral group consists of symmetries inherent to a regular n-polygon, and it adheres to the principles outlined by group axioms. The formal definition is given below.

**Definition 1.** [6] The dihedral group, denoted by $D_{2n}$ is the set:

$$D_{2n} = \{e, a, a^2, \ldots, a^{n-1}, b, ab, \ldots, a^{n-1}b \mid a^n = b^2 = e, a^{-1} = bab\}, n \geq 3.$$

A graph is a mathematical structure widely employed in various fields such as computer science, network modeling, and others. The graph is defined as follows:

**Definition 2.** A graph is a mathematical structure consisting of a non-empty set of vertices and a set of edges connecting these vertices. A graph is said to be complete if every vertex is connected with an edge.

The non-coprime graph constitutes a novel method for graph representation, and it was introduced by Mansoori [7]. A non-coprime graph involves the order of the group elements as described in the following definition.

**Definition 3.** [7] Let $G$ is a finite group, the non-coprime graph of group $G$ denoted by $\Gamma_G$ is a graph with vertices consisting of $\bar{G} = G - \{e\}$ and two different vertices $x, y \in \bar{G}$ is adjacent whenever $(\text{ord}(x), \text{ord}(y)) \neq 1$. For every $x \in \bar{G}$, the $\text{ord}(x)$ is the smallest positive integer $k$ such that $x^k = e$.

In this section, we will discuss the shape of non-coprime graphs in dihedral groups. The non-coprime graph for the dihedral group $D_6$ shown in Table 1. Figure 1 shows the non-coprime graph from the dihedral group $D_6$. Based on Figure 1, we discovered from the non-coprime graph from the dihedral group $D_6$ is a combination of two complete graphs $K_2 \cup K_3$. 
The Shape of a Non-Coprime Graph of a Dihedral Group

Table 1. Order from the dihedral group $D_6$

<table>
<thead>
<tr>
<th>Component</th>
<th>Order</th>
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<tbody>
<tr>
<td>$e$</td>
<td>1</td>
</tr>
<tr>
<td>$a$</td>
<td>3</td>
</tr>
<tr>
<td>$a^2$</td>
<td>3</td>
</tr>
<tr>
<td>$b$</td>
<td>2</td>
</tr>
<tr>
<td>$ab$</td>
<td>2</td>
</tr>
<tr>
<td>$a^2b$</td>
<td>2</td>
</tr>
</tbody>
</table>

Figure 1. Non-coprime graph from dihedral group $D_6$

Next, the non-coprime graph from the dihedral group $D_8$ shown in Table 2. Figure 2 shows a non-coprime graph from the dihedral group $D_8$. Based on Figure 2, we discovered the shape of the non-coprime graph from the dihedral group $D_8$ is a complete graph $K_7$.

Table 2. Order from the dihedral group $D_8$

<table>
<thead>
<tr>
<th>Component</th>
<th>Order</th>
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<tbody>
<tr>
<td>$e$</td>
<td>1</td>
</tr>
<tr>
<td>$a$</td>
<td>4</td>
</tr>
<tr>
<td>$a^2$</td>
<td>2</td>
</tr>
<tr>
<td>$a^3$</td>
<td>4</td>
</tr>
<tr>
<td>$b$</td>
<td>2</td>
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<tr>
<td>$ab$</td>
<td>2</td>
</tr>
<tr>
<td>$a^2b$</td>
<td>2</td>
</tr>
<tr>
<td>$a^3b$</td>
<td>2</td>
</tr>
</tbody>
</table>

Next, the non-coprime graph from the dihedral group $D_{10}$ shown in Table 3. Figure 3 shows a non-coprime graph from the dihedral group $D_{10}$. Based on Figure 3, we discovered the shape of the non-coprime graph from the dihedral group $D_{10}$ is the combination of two complete graphs $K_4 \cup K_5$. Using a similar method, we discover the shape of the non-coprime of the dihedral groups $D_{14}, D_{16},$ and $D_{18}$ as well.
Figure 2. Non-coprime graph from dihedral group $D_8$

Table 3. Order from the dihedral group $D_{10}$

<table>
<thead>
<tr>
<th>Component</th>
<th>Order</th>
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</thead>
<tbody>
<tr>
<td>$e$</td>
<td>1</td>
</tr>
<tr>
<td>$a$</td>
<td>5</td>
</tr>
<tr>
<td>$a^2$</td>
<td>5</td>
</tr>
<tr>
<td>$a^3$</td>
<td>5</td>
</tr>
<tr>
<td>$a^4$</td>
<td>5</td>
</tr>
<tr>
<td>$b$</td>
<td>2</td>
</tr>
<tr>
<td>$ab$</td>
<td>2</td>
</tr>
<tr>
<td>$a^2b$</td>
<td>2</td>
</tr>
<tr>
<td>$a^3b$</td>
<td>2</td>
</tr>
<tr>
<td>$a^4b$</td>
<td>2</td>
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</table>

Figure 3. Non-coprime graph from dihedral group $D_{10}$

Figure 4 shows a non-coprime graph from the dihedral group $D_{14}$. Based on Figure 4, we discovered the shape of the non-coprime graph from the dihedral group $D_{14}$ is a combination of two complete graphs $K_6 \cup K_7$. Figure 5 shows a non-coprime graph from the dihedral group $D_{16}$. Based on this figure, we discovered the shape of the non-coprime graph from the dihedral group $D_{16}$ is a complete graph $K_{15}$.
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Figure 4. Non-coprime graph from dihedral group $D_{14}$

Figure 5. Non-coprime graph from the dihedral group $D_{16}$

Figure 6 shows a non-coprime graph from the dihedral group $D_{18}$. Based on Figure 6, we discovered the shape of the non-coprime graph from the dihedral group $D_{18}$ is a combination of two complete graphs $K_8 \cup K_9$. And with several examples of the shapes of non-coprime graphs obtained previously, we obtained several theorems related to the characteristics of non-coprime graphs of dihedral groups, which is more general than what Misuki gave [20].

Figure 6. Non-coprime graph from dihedral group $D_{18}$
Misuki, in his study, already found interesting properties of non-coprime graphs in the following results.

**Theorem 1.** [4] Given $D_{2n}$ as the dihedral group. If $n = 2^k$ for some $k \in \mathbb{N}$, then $\Gamma_{D_{2n}}$ is a complete graph.

Apart from the above properties, a non-coprime graph in a dihedral group where $n$ is a prime number has other properties stated in the following theorem.

**Theorem 2.** [4] Given $D_{2n}$ as the dihedral group. If $n = p^k$ for some $k \in \mathbb{N}$, and $p$ is an odd prime. Then $\Gamma_{D_{2n}}$ consist of two disjoint complete graphs.

The last property given by Misuki in his study related to non-coprime graphs in the dihedral group where $n$ is a prime number is described in the following theorem.

**Theorem 3.** [4] Given $H$ as a non-trivial subgroup of the dihedral group $D_{2n}$. If $n = p^m$ then the non-coprime graph of $H$ is a trivial graph or a complete graph or consist of two complete graphs.

Based on the result of the non-coprime graph given by Misuki, we give the more general result of the structure of the non-coprime graph in the dihedral group. Initially, it was observed that it is always possible to identify a subgraph within which forms a complete graph.

**Theorem 4.** Given $D_{2n}$ is a dihedral group. If $n = p_1^{k_1}p_2^{k_2}...p_m^{k_m}$, where $p_1, p_2, ..., p_m$ are distinct odd primes, then $\Gamma_{D_{2n}}$ has a complete subgraph.

**Proof:** Let $G$ a subset of $D_{2n} - \{e\}$, where $G = \{b, ab, a^2b, ..., a^{n-1}b\}$, because $G$ consists of all reflection elements; consequently, 2 divides each order of $y \in G$, so that each element in $G$ is neighboring, consequently $\Gamma_{D_{2n}}$ has a complete subgraph. ■

Second, we give a more general result of Theorem 2.

**Theorem 5.** Given $D_{2n}$ is a dihedral group. If $n = p_1^{k_1}p_2^{k_2}...p_m^{k_m}$, where $p_1, p_2, ..., p_m$ are distinct odd prime and $k_1, k_2, ..., k_m \in \mathbb{N}$, then $\Gamma_{D_{2n}}$ has $m + 1$ disjoint complete subgraph.

**Proof:** Define $m + 1$ set, the first set $V_1$ is a set of elements with order 2, and the second set $V_2$ is a set of all elements in order $a_1$, such that $p_1|a_1$. And for $i = 3, ..., m$, the $i$-set $V_i$ is a set of an element with odd order $o_{i-1}$, such that $p_{i-1}|o_{i-1}$ and $p_j$ not divide $o_{i-1}$ for all $1 \leq j \leq i - 2$. Noted that, since $2 \notin n$, all elements in $V_j$ have an odd order for $j > 1$. Let $x \in V_i$ and $y \in V_j$, without loss of generality, let's assume that $1 < i < j$. We have $p_{i-1}|\text{ord}(x)$ and $p_i \nmid \text{ord}(x)$ for all $l < i - 1$ and also $p_j|\text{ord}(y)$ and $p_r \nmid \text{ord}(y)$ for all $r < j - 1$, thus $(\text{ord}(x), \text{ord}(y)) = 1$, then $x$ and $y$ are not adjacent. So, we can conclude that $D_{2n}$ has $m + 1$ disjoint component.

Now we need to prove that every component is a complete component. Note that $V_1$ is a complete graph by Theorem 4. And for $i > 1$, every $x, y \in V_i$, we have $p_{i-1}|\text{ord}(x)$ and $p_{i-1}|\text{ord}(y)$, so $x, y$ are adjacent. Hence $V_i$ is a complete component. ■

We found a sufficient condition for the non-coprime graph to be connected.

**Theorem 6.** Given $D_{2n}$ is a dihedral group. If $2|n$ then $\Gamma_{D_{2n}}$ is a connected graph.
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Proof: Let $x, y \in D_{2n} - \{e\}$ arbitrary and $n = p_1^{k_1} p_2^{k_2} \ldots p_m^{k_m}$ where $p_1, p_2, \ldots, p_m$ are distinct odd prime and $k_1, k_2, \ldots, k_m \in \mathbb{N}$. We can assume that $p_i | \text{ord}(x)$ and $p_j | \text{ord}(y)$ for some $i, j \in \mathbb{N}$, then there exists $a, b \in D_{2n} - \{e\}$ such that $\text{ord}(a) = 2p_i$ and $\text{ord}(b) = 2p_j$. Note that there is always an element of $D_{2n}$ with order 2, named it $c$. Then we have a path $x - a - c - b - y$, since $x, y$ arbitrary, then $\Gamma_{D_{2n}}$ is a connected graph.

Now we can give the direct result of Theorem 5. If 2 does not divide $n$, then the non-coprime graph $\Gamma_{D_{2n}}$ is not connected.

Corollary 2. Given $D_{2n}$ is a dihedral group. If $n = p_1^{k_1} p_2^{k_2} \ldots p_m^{k_m}$, and $m \geq 2$ where $p_1, p_2, \ldots, p_m$ are distinct odd prime and $k_1, k_2, \ldots, k_m \in \mathbb{N}$, then $\Gamma_{D_{2n}}$ is not a connected graph.

4. CONCLUSIONS

Based on this study, we can conclude that a non-coprime graph of the dihedral group has a structure that consists of some disjoint complete subgraph.

REFERENCES


