M. Adib Jauhari Dwi Putra* and Ade Ima Afifa Himayati  
Department of Mathematics, Universitas Muhammadiyah Kudus, Jl Ganesha I Purwosari Kudus Indonesia  
Email: adibjauhari@umkudus.ac.id  

Abstract  
We studied the Leslie-Gower model of predator-prey with herd behavior. The square root functional response models predator and prey interactions that show herd behavior. This study aims to determine the formulation of the predator-prey model with herd behavior on prey, knowing the fixed points and its stability and simulating the model numerically. We found three fixed points that may exist: the extinction point of both species, the extinction of predator point, and the point of coexistence of the two species. The extinction of predator points is always unstable, while the point of coexistence of the two species can be stable under certain conditions. Due to the presence of square roots, the behavior of the solutions near the extinction point of the two species is not readily apparent. Numeric simulation shows that changing the initial condition and parameters can change the system's stability.  
Keywords: predator-prey; functional response; herd behavior; square root functional response, Leslie-Gower model.  

1. INTRODUCTION  
Predator-prey model is an essential tool for understanding the interactions between species in a natural environment. The first model describing the interactions between two species, prey, and predator, was introduced by Lotka and Volterra, which became known as the Lotka-Volterra predator-prey model [1] [2] [3]. The model is a system of ordinary differential equations. The predation function of predators on prey in this model is linear or called Holling type I. Other common functional responses for predator-prey models are types II, III, and IV [4] [5] [6] [7]. Leslie and Gower modified the model by adding carrying capacity for the predator population in proportion to the size or number of the prey population [8]. The predator-prey model often studied is the Leslie-Gower model with a linear response function as in the Collings [9]. This model is:
\[
\begin{align*}
\frac{dX}{dt} &= r_1 X \left(1 - \frac{X}{K}\right) - aXY, \\
\frac{dY}{dt} &= r_2 Y \left(1 - \frac{Y}{bX}\right),
\end{align*}
\]

where \( r_1, r_2, a, b, K > 0 \), \( r_1 \) and \( r_2 \) represent the intrinsic growth of predator and prey, respectively, \( a \) describes the rate of predation, and \( b \) describes the growth rate of the predator as a result of predation. Aziz-Alaoui and Okiye [10] modified Leslie Gower's predator-prey model by adding the amount of environmental protection to maintain the survival of predators. The model is as follows:

\[
\begin{align*}
\frac{dX}{dt} &= r_1 X \left(1 - \frac{X}{K}\right) - \frac{aX}{m_1 + X} Y, \\
\frac{dY}{dt} &= r_2 Y \left(1 - \frac{Y}{m_2 + X}\right),
\end{align*}
\]

where \( m_1, m_2, r_1, r_2, a, K > 0 \), \( m_1 \) and \( m_2 \) describe the environmental protection of predator and prey populations, respectively.

Animals often have herd behavior as a form of self-defense from predators. As a result, predators can only attack prey outside the pack. As in animals in a savanna, predators can only attack the outside of the swarming prey group. Or the fish that cluster in the sea to defend themselves from shark attacks. That social behavior in the prey population often appears because they live in a herd. The individuals living on the outskirts can protect those residing at the flock's center. Such a model is considered by Ajraldi et al. [11], assuming that the interactions occur along the boundary of the population. They use the square root of the prey population in interaction terms to describe herd behavior.

Braza [12] also described herd behavior in prey with the square root response function. The proposed model is to modify the Lotka-Volterra model using the Holling type II and square root functional responses as follows:

\[
\begin{align*}
\frac{dX}{dt} &= r_1 X \left(1 - \frac{X}{K}\right) - \frac{a\sqrt{X}Y}{1 + \sqrt{X}}, \\
\frac{dY}{dt} &= -sY + \frac{b\sqrt{XY}}{1 + a\sqrt{X}},
\end{align*}
\]

where \( s \) is the death rate of the predator. He has shown that the square root term makes the solution behavior near the origin more subtle than standard models. In this article, we studied the Leslie-Gower model of predator-prey with herd behavior. We determine the formulation of the predator-prey model with herd behavior on prey, knowing the fixed points and their stability. In addition, at the end of the article, we present a numerical simulation of the model.

2. METHODS

We modified the two-species Leslie-Gower model of the system (1) by adding square root functional response as in system (3) to describe herd behavior in prey. So that the model becomes
where each parameter is positive. Table 1 shows the variables and parameters used in our model.

### Table 1: Model variables and parameters

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X$</td>
<td>Number of prey</td>
</tr>
<tr>
<td>$Y$</td>
<td>Number of predators</td>
</tr>
<tr>
<td>$t$</td>
<td>Time</td>
</tr>
<tr>
<td>$r_1$</td>
<td>Intrinsic growth of prey</td>
</tr>
<tr>
<td>$r_2$</td>
<td>Intrinsic growth of predator</td>
</tr>
<tr>
<td>$a$</td>
<td>Rate of predation</td>
</tr>
<tr>
<td>$b$</td>
<td>The growth rate of the predator as a result of predation</td>
</tr>
<tr>
<td>$K$</td>
<td>Carrying capacity</td>
</tr>
</tbody>
</table>

First, we calculate the system's fixed points and then perform a local stability analysis of those points by linearization [13]. The next step is to do some numerical simulations using the Maple program to see the system's behavior by changing a parameter and initial values. Finally, we explain the results of the numerical simulations and conclude the system.

3. **RESULTS AND DISCUSSIONS**

3.1 Fixed Points of the System and Its Stability

The system's fixed point (4) is obtained by solving the equation $\frac{dX}{dt} = 0$ and $\frac{dY}{dt} = 0$. There are three fixed-point solutions:

1. $E_1 = (0,0)$, which illustrates that both species are extinct,
2. $E_2 = (K,0)$, which illustrates that only the prey species have survived,
3. $E_3 = \left( \frac{K(r_1-ab)}{r_1}, \frac{bK(r_1-ab)}{r_1} \right)$ where $K - \frac{Kab}{r_1} > 0$, a fixed point where prey and predator coexist.

The stability of this fixed point determine using the eigenvalues of the following Jacobian matrix,

$$
\begin{bmatrix}
    r_1 - \frac{2r_1X}{K} - \frac{aY}{2\sqrt{X}} & -a\sqrt{X} \\
    \frac{r_2Y^2}{2bX\sqrt{X}} & r_2 - \frac{2r_2Y}{b\sqrt{X}}
\end{bmatrix}.
$$

This Jacobian matrix is evaluated for each fixed point. For point $E_1 = (0,0)$ the characteristic equation cannot be determined because the values of $\frac{r_2Y^2}{2bX\sqrt{X}}$ for $X = 0$ and $Y = 0$ are undefined. The system behavior around point $E_1$ will be explained in the numerical simulation. Furthermore, the system behavior in $E_2$ and $E_3$ are as follows.
1. A fixed point where only prey species exist

The Jacobian matrix at point $E_2$ is

$$J(E_2) = \begin{bmatrix} -r_1 & -a\sqrt{K} \\ 0 & r_2 \end{bmatrix}$$

The eigenvalues of $J(E_2)$ are $\lambda_1 = -r_1$ and $\lambda_2 = r_2$, which are the saddle points. Since each parameter is positive, point $E_2$ is always unstable.

2. A fixed point where predator and prey co-exist

The Jacobian matrix at point $E_3$ is

$$J(E_3) = \begin{bmatrix} -r_1 + \frac{3}{2}ab & -a \sqrt{\frac{K(r_1 - ab)}{r_1}} \\ \frac{r_2 b}{2\sqrt{K r_1 - ab}} & r_2 \end{bmatrix}$$

The characteristic equation of the matrix $J(E_3)$ is $\lambda^2 + \left(r_1 + r_2 - \frac{3}{2}ab\right)\lambda + r_1 r_2 - ab r_2$. So that the eigenvalue of $J(E_3)$ is

$$\lambda_{1,2} = \frac{-\left(r_1 + r_2 - \frac{3}{2}ab\right) \pm \sqrt{D}}{2},$$

with $D = \left(r_1 + r_2 - \frac{3}{2}ab\right)^2 - 4(r_1 r_2 - ab r_2)$. Point $E_3$ is stable if both eigenvalues are negative, that is, when $D \geq 0$, $\lambda_1 + \lambda_2 < 0$, and $\lambda_1 \cdot \lambda_2 > 0$.

3.2 Numerical Simulations

This section presents some numerical simulation results that illustrate the system's behavior (4) by varying the system's parameters. In Figure 1, we use the parameters $r_1 = 0.8; r_2 = 0.3; K = 5; a = 0.5$ and $b = 1$. Three fixed points exist i.e. the point of extinction for the two species $E_1$, which is unstable; the point of extinction for predator $E_2$, which is also unstable; and the point $E_3$ which shows the two species coexist, which is asymptotically stable. Points $E_1$ and $E_2$ can be saddle points because each trajectory approaches the two points and then moves away at a certain initial value, while point $E_3$ is a spiral sink.

Furthermore, using the parameters $r_1 = 2; r_2 = 0.5; K = 5; a = 0.7; \text{ and } b = 1$. The system behavior also shows results similar to Figure 2. There are three points that exist: the unstable extinction point of the two species, the unstable predator extinction point, and the coexistence of the two species point that is asymptotically stable.

In the third simulation, the parameters $r_1 = 0.5; r_2 = 0.8; K = 0.5; a = 0.8, \text{ and } b = 0.7$. In Figure 3, it can be seen that only 2 equilibrium points exist because the value of $K - \frac{Kab}{r_1} = -0.06 < 0$, so $E_3$
does not exist. The extinction point of the two species $E_1$ is stable, and the predator extinction point $E_2$ is unstable.

**Figure 1.** There are three equilibrium points that exist, $E_1$, $E_2$, and $E_3$.

**Figure 2.** Parameter change shifts point $E_3$ but do not affect its stability.

In Figure 4, the parameters used are $r_1 = 0.5; r_2 = 0.01; K = 5; a = 0.8$, and $b = 0.05$. There are three fixed points in existence. The stability of points $E_1$ and $E_3$ depends on the starting point or initial value used. While pointing out $E_2$ is unstable.
4. CONCLUSIONS

This paper studied the Leslie-Gower model with a square root functional response that describes herd behavior on prey. We have shown that three equilibrium points exist, $E_1$, $E_2$, and $E_3$. We are also shown that $E_2$ is always unstable and $E_3$ will be stable if $\left( r_1 + r_2 - \frac{3}{2}ab \right)^2 - 4(r_1r_2 - abr_2) \geq 0$, $\lambda_1 + \lambda_2 < 0$ and $\lambda_1 \cdot \lambda_2 > 0$. The stability of $E_1$ cannot be determined by linearization. Based on the numeric simulation, the stability of $E_1$ depends on the initial condition. We have observed that the square root functional response can make the system near the origin more subtle. If the prey
population is smaller than the predator, both species will reach extinction, and if the predator has a smaller population, the system will be away from the origin. This makes an ecological balance. Herd behavior on prey is like self-defense to avoid predation. The predator can only attack the outside of the herd.

In this study, we have analyzed the stability of the system locally. For further research, we suggest analyzing the occurrence of bifurcation in the system and its stability globally. This system modification is also implemented by changing the system to add other variables so that it becomes 3 species system.

REFERENCES