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Fair Secure Roman Dominating Function in Graphs

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(*This paper is dedicated to our Professor Sergio R. Canoy Jr.*)

Abstract

Let $G = (V(G), E(G))$ be a graph and let $\phi: V(G) \to \{0,1,2\}$ be a function on G. For each $i \in \{0,1,2\}$, let $V_i = \{v \in V(G): \phi(v) = i\}$. Then ϕ can be represented as $\phi = (V_0, V_1, V_2)$. A function ϕ is a fair secure Roman dominating function (FScRDF) on G provided that for every $v \in V_0$, there exists $u \in V_2$ such that $d_G(u,v) = 1$, $\phi^* = (V_0 \setminus \{v\}, V_1 \cup \{v,u\}, V_2 \setminus \{u\})$ is a Roman dominating function (RDF) on G, and for every $x, y \in V_0$, $| N_G(x) \cap V_2 | = | N_G(y) \cap V_2 | \geq 1$. The weight of FScRDF ϕ on G, denoted by $\omega_G^{FScR}(\phi)$, is defined as the sum $\omega_G^{FScR}(\phi) = \sum_{x \in V(G)} \phi(x) = | V_1 | + 2 | V_2 |$. The fair secure Roman domination number of G is defined as the minimum weight of an FScRDF ϕ on G, and is denoted by $\gamma_{FScR}(G)$, that is, $\gamma_{FScR}(G) = min\,\{\omega_G^{FScR}(\phi): \phi \text{ is an FScRDF on } G\}$. Every FScRDF ϕ on G that satisfies $\omega_G^{FScR}(\phi) = \gamma_{FScR}(G)$ is called a γ_{FScR}-function on G. In this paper, the authors introduce the idea of fair secure Roman domination in graphs as a new parameter and discuss some important combinatorial results.

Keywords: Fair domination; Fair secure Roman dominating function; Fair secure Roman domination number; Roman dominating function; Secure domination.

Abstrak

Misalkan $G = (V(G), E(G))$ *adalah graf dan* $\phi: V(G) \to \{0,1,2\}$ *adalah fungsi di* G. *Untuk setiap* $i \in \{0,1,2\}$, *misalkan* $V_i = \{v \in V(G): \phi(v) = i\}$. *Fungsi* ϕ *dapat disajikan dalam bentuk* $\phi = (V_0, V_1, V_2)$. *Fungsi* ϕ *dikatakan suatu fungsi mendominasi Roman aman cukup (fair secure Roman dominating function; FScRDF) di* G *apabila untuk setiap* $v \in V_0$, *terdapat* $u \in V_2$ *sehingga* $d_G(u,v) = 1$, $\phi^* = (V_0 \setminus \{v\}, V_1 \cup \{v,u\}, V_2 \setminus \{u\})$ *adalah suatu fungsi mendominasi Roman (Roman dominating function; RDF) di* G, *dan untuk setiap* $x, y \in V_0$, $| N_G(x) \cap V_2 | = | N_G(y) \cap V_2 | \geq 1$. *Bobot dari FScRDF* ϕ *di* G, *dinotasikan dengan* $\omega_G^{FScR}(\phi)$, *didefinisikan sebagai jumlah* $\omega_G^{FScR}(\phi) = \sum_{x \in V(G)} \phi(x) = | V_1 | + 2 | V_2 |$. *Bilangan dominasi Roman aman cukup dari* G *didefinisikan sebagai bobot minimum dari suatu FScRDF* ϕ *di* G, *dan dinotasikan dengan* $\gamma_{FScR}(G)$, *yakni,* $\gamma_{FScR}(G) = min\,\{\omega_G^{FScR}(\phi): \phi \text{ adalah suatu FScRDF di } G\}$. *Setiap FScRDF* ϕ *di* G *yang memenuhi* $\omega_G^{FScR}(\phi) = \gamma_{FScR}(G)$ *disebut suatu fungsi-*γ_{FScR} *di* G. *Dalam paper ini, penulis memperkenalkan gagasan dominasi Roman aman cukup pada graf sebagai suatu parameter baru dan mendiskusikan beberapa hasil kombinatorial penting.*

Kata Kunci: *Dominasi cukup; Fungsi mendominasi Roman aman cukup, Bilangan dominasi Roman aman cukup, Fungsi mendominasi Roman, Dominasi aman.*

2020MSC: 05C69

1. INTRODUCTION

Roman domination in graphs was pioneered by Cockayne et al. [1] in the year 2004, and its concept is founded on a historical defense strategy in the Roman Empire during the fourth century A.D. Roman domination has now become a famous parameter in domination theory in graphs and is considered an interesting topic, for which many discrete mathematicians are contributing papers that

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involve the said concept. Some interesting papers that involve Roman domination in graphs are found in [2, 3, 4]. In 2003, the paper of Cockayne et al. [5] introduced a new restricted variation of domination in graphs called secure domination, and in 2012, Caro et al. [6] initiated the concept of fair dominating set in graphs. Recently, Enriquez [2][7], during the year 2020, combined the concepts of fair domination and secure domination and came up with an interesting parameter called fair secure domination in graphs. Henceforth, the author is inspired to combine the idea of fair secure domination and the Roman dominating function and come up with a new restricted parameter in domination theory. The purpose of this study is to address the shortcomings of existing Roman domination in graphs and strengthen its strategic defense, thereby contributing to the body of knowledge in graph theory.

2. DEFINITIONS

For the terminologies and definitions of the concepts in graph theory used in this paper, the readers may refer to [8, 9]. Let $G = (V(G), E(G))$ be a simple and finite graph of order $n \geq 1$. Let D be a subset of $V(G)$. Then set D is called a *dominating set* of G provided that for every $V(G) \setminus D$ is adjacent to some vertex in D [10]. The *domination number* of G is denoted by $\gamma(G)$ and is defined as the minimum cardinality of the set D on G. If $|D| = \gamma(G)$, then a set D is called a γ-set on G. For some studies on dominating set in graphs, the readers may refer to [11, 12, 13, 14, 15]. Let F be a proper subset of $V(G)$. Then F is called a *fair vertex set* on G provided that for any $x, y \in F$, $|N_G(x) \cap (V(G) \setminus F)| = |N_G(y) \cap (V(G) \setminus F)| > 0$. Let D_f be a subset of $V(G)$. Then D_f is called a *fair dominating set* of G if every vertex in $V(G) \setminus D_f$ is adjacent to some vertex in D_f and the set $V(G) \setminus D_f$ is a fair vertex set on G, that is, for every $x, y \in V(G) \setminus D_f$, $|N_G(x) \cap D_f| = |N_G(y) \cap D_f| > 0$ [6]. The *fair domination number* of G is denoted by $fd(G)$ and is defined as the smallest cardinality of F_d on G. If $|D_f| = fd(G)$, then the set D_f is called an *FD-set* on G. Let S be a subset of $V(G)$. Then $S$is called a *secure dominating set* of G if for every $v \in V(G) \setminus S$ there exists $u \in S$ such that $d_G(u, v) = 1$ and the set $S \setminus \{u\} \cup \{v\}$ is a dominating set on G [5]. The *secure domination number* of G is denoted by $\gamma_S(G)$ and is defined as the smallest cardinality of S on G. If $|S| = \gamma_S(G)$, then the set S is called a γ_S-set on G. Let F_S be a subset of $V(G)$. Then F_S is called a *fair secure dominating set* of G provided that for every $v \in V(G) \setminus F_S$ there exists $u \in F_S$ such that $d_G(u, v) = 1$, the set $F_S \setminus \{u\} \cup \{v\}$ is a dominating set on G, and for every $x, y \in V(G) \setminus F_S$, $|N_G(x) \cap F_S| = |N_G(y) \cap F_S| \geq 1$ [7]. The *fair secure domination number* of G is denoted by $\gamma_{fS}(G)$ and is defined as the smallest cardinality of F_S on G. If $|F_S| = \gamma_{fS}(G)$, then the set F_S is called a γ_{fS}-set on G.

Let $\phi: V(G) \to \{0,1,2\}$ be a function on G. If we let $V_i = \{u \in V(G): \phi(u) = i\}$, for each $i \in \{0,1,2\}$, then ϕ can be represented as $\phi = (V_0, V_1, V_2)$. A function $\phi = (V_0, V_1, V_2)$ is called a *Roman dominating function (RDF)* on G if for every vertex $v \in V_0$ there exists $u \in V_2$ such that $d_G(u, v) = 1$ [1]. The weight of function ϕ on G, denoted by $\omega_G^R(\phi)$, is defined by $\omega_G^R(\phi) = \sum_{w \in V(G)} \phi(w) = |V_1| + 2|V_2|$. The *Roman domination number* of G, denoted by $\gamma_R(G)$, is defined as the minimum weight of an RDF on G, that is, $\gamma_R(G) = \min\{\omega_G^R(\phi): \phi \text{ is an RDF on } G\}$. Moreover, every RDF ϕ on G with $\omega_G^R(\phi) = \gamma_R(G)$ is called a γ_R-function on G. A function ϕ is a *fair secure Roman dominating function*

(FScRDF) on G provided that for every $v \in V_0$, there exists $u \in V_2$ such that $d_G(u,v) = 1$, $\phi^* = (V_0 \setminus \{v\}, V_1 \cup \{v,u\}, V_2 \setminus \{u\})$ is an RDF on G, and for every $x,y \in V_0$, $| N_G(x) \cap V_2 |=| N_G(y) \cap V_2 | \geq 1$. The weight of FScRDF ϕ on G, denoted by $\omega_G^{FScR}(\phi)$, is defined as the sum $\omega_G^{FScR}(\phi) = \sum_{x\in V(G)} \phi(x) =| V_1 | +2 | V_2 |$. The *fair secure Roman domination number* of G is defined as the minimum weight of an FScRDF ϕ on G, that is, $\gamma_{FScR}(G) = \min\{\omega_G^{FScR}(\phi) : \phi \text{ is an FScRDF on } G\}$, and is denoted by $\gamma_{FScR}(G)$. Every FScRDF ϕ on G that satisfies $\omega_G^{FScR}(\phi) = \gamma_{FScR}(G)$ is called a γ_{FScR}-function on G. In this paper, we introduce the idea of fair secure Roman domination in graphs as a new Roman domination parameter and provide some important combinatorial and graph-theoretic findings.

3. RESULT

This section explores the combinatorial properties of a fair secure Roman dominating function and characterizes the function on some classes of graphs.

Theorem 1. Let G be any graph and let $\phi = (V_0, V_1, V_2)$ be a FScRDF on G. If for each $u \in V_2$, $| N_G(u) \cap V_0 | \geq 2$, then for any $x,y \in V_0$, $| N_G(x) \cap V_2 |=| N_G(y) \cap V_2 | \geq 2$.

Proof.
Let $\phi = (V_0, V_1, V_2)$ be an FScRDF on G. Suppose that for each $u \in V_2$, $| N_G(u) \cap V_0 | \geq 2$. Assume for a moment that for any $x,y \in V_0$, $| N_G(x) \cap V_2 |=| N_G(y) \cap V_2 | \leq 1$. If $| N_G(x) \cap V_2 |=| N_G(y) \cap V_2 | = 0$, then it means that V_2 is not a dominating set on G. This contradicts the definition of RDF on G. Now, consider $| N_G(x) \cap V_2 |=| N_G(y) \cap V_2 |= 1$. Let $w \in N_G(x) \cap V_2$. Then $w \in V_2$. By assumption, it follows that $| N_G(w) \cap V_0 | \geq 2$. This implies that there exists $z \in V_0 \setminus \{x\}$ such that $d_G(z,w) = 1$. Now, consider the function $\phi^* = (V_0 \setminus \{z\}, V_1 \cup \{w,z\}, V_2 \setminus \{w\})$ on G. However, $N_G(x) \cap (V_2 \setminus \{w\}) = \emptyset$, implying that ϕ^* is not an RDF on G, a contradiction. Therefore, we conclude that for any $x,y \in V_0$, $| N_G(x) \cap V_2 |=| N_G(y) \cap V_2 | \geq 2$. This completes the proof. ■

Theorem 2. Let G be a graph with $| V(G) |= n$ and let $\phi = (V_0, V_1, V_2)$ be a γ_{FScR}-function on G. If for each $u \in V_2$, $| N_G(u) \cap V_0 |= 1$, then for any $x,y \in V_0$, $| N_G(x) \cap V_2 |=| N_G(y) \cap V_2 |= 1$. In that case, $| V_0 |=| V_2 |$ if and only if $\gamma_{FScR}(G) = n$.

Proof.
Let $\phi = (V_0, V_1, V_2)$ be a γ_{FScR}-function on G of order n. Suppose for each $u \in V_2$, $| N_G(u) \cap V_0 |= 1$. Assume for a moment that for any $x,y \in V_0$, $| N_G(x) \cap V_2 |=| N_G(y) \cap V_2 | \neq 1$. If $| N_G(x) \cap V_2 |=| N_G(y) \cap V_2 |= 0$, then V_2 is not a dominating set on G, a contradiction to the definition of RDF on G. If $| N_G(x) \cap V_2 |=| N_G(y) \cap V_2 | \geq 2$, then it follows that there exists $w \in V_2$ such that $| N_G(w) \cap V_0 | > 1$. This is a contradiction to our assumption. Thus, for any $x,y \in V_0$, $| N_G(x) \cap V_2 |=| N_G(y) \cap V_2 |= 1$.

Now, since for every $v \in V_2$, $\mid N_G(v) \cap V_0 \mid= 1$, it implies that the mapping $f: V_2 \to V_0$ is a one-to-one and onto, and thus, $\mid V_2 \mid=\mid V_0 \mid$. In that case, we obtain $\gamma_{FScR}(G) = \omega_G^{FScR}(\phi) =\mid V_1 \mid +2 \mid V_2 \mid=\mid V_0 \mid +\mid V_1 \mid +\mid V_2 \mid=\mid V(G) \mid= n$. Conversely, suppose $\gamma_{FScR}(G) = n$. Assume for a moment that $\mid V_0 \mid\neq\mid V_2 \mid$. Then either $\mid V_0 \mid>\mid V_2 \mid$ or $\mid V_0 \mid<\mid V_2 \mid$. If $\mid V_0 \mid>\mid V_2 \mid$, then $\gamma_{FScR}(G) = \omega_G^{FScR}(\lambda) =\mid V_1 \mid +2 \mid V_2 \mid<\mid V_0 \mid +\mid V_1 \mid +\mid V_2 \mid=\mid V(G) \mid= n$, a contradiction. If $\mid V_0 \mid<\mid V_2 \mid$, then $\gamma_{FScR}(G) = \omega_G^{FScR}(\lambda) =\mid V_1 \mid +2 \mid V_2 \mid >\mid V_0 \mid +\mid V_1 \mid +\mid V_2 \mid=\mid V(G) \mid= n$, a contradiction. Therefore, we conclude that $\mid V_0 \mid=\mid V_2 \mid$. This completes the proof. ■

The corollaries below are quick from Theorem 2.

Corollary 3. Let G be a graph and let $\phi = (V_0, V_1, V_2)$ be an FScRDF on G. If for each $v \in V_0$, $\mid N_G(v) \cap V_2 \mid= 1$, then there exists $u \in V_2$ such that $N_G(u) \cap V_0 = \{v\}$.

Corollary 4. Let G be a graph and let $\phi = (V_0, V_1, V_2)$ be a γ_{FScR}-function on G. Then $\gamma_{FScR}(G) < n$ if and only if $\mid V_2 \mid<\mid V_0 \mid$.

Theorem 5. Let G be a graph with $\mid V(G) \mid= n$ and let $\phi = (V_0, V_1, V_2)$ be an FScRDF on G. Then (i) $V_1 \cup V_2$ is a fair secure dominating set on G; and (ii) ϕ is a γ_{FScR}-function on G implies that $V_0 = \emptyset$ if and only if $V_2 = \emptyset$. In that case, $\gamma_{FScR}(G) = n$.

Proof.
Let $\phi = (V_0, V_1, V_2)$ be an FScRDF on G. Then it follows that for each $v \in V_0$, there exists $u \in V_2$ such that $uv \in E(G)$, that is, $V_0 \subseteq N_G[V_2]$, $\phi_v = (V_0 \setminus \{v\}, V_1 \cup \{v, u\}, V_2 \setminus \{u\})$ is an RDF on G, and for every $x, y \in V_0$, $\mid N_G(x) \cap V_2 \mid=\mid N_G(y) \cap V_2 \mid\geq 1$. Let $S = [(V_1 \cup V_2) \setminus \{u\}] \cup \{v\}$ and let $w \in V(G) \setminus S$. Then $w \in (V_0 \setminus \{v\}) \cup \{u\}$. Suppose $w = u$. Then $v \in S \cap N_G(w)$. Moreover, we suppose that $w \neq u$. Then $w \in V_0 \setminus \{v\}$. Since ϕ_v is an RDF on G, there exists $a \in (V_2 \setminus \{u\}) \cap N_G(w)$. Hence, $a \in S \cap N_G(w)$. Consequently, S is a dominating set on G. Therefore, $V_1 \cup V_2$ is a secure dominating set on G. Since for every $x, y \in V_0$, $\mid N_G(x) \cap V_2 \mid=\mid N_G(y) \cap V_2 \mid\geq 1$, it means that $V_1 \cup V_2$ is a fair secure dominating set on G. Thus, (i) holds. Next, we let ϕ be a γ_{FScR}-function on G. Suppose that $V_0 = \emptyset$. Assume for a moment that $V_2 \neq \emptyset$. Let $u \in V_2$. Then set $U_0 = V_0$, $U_1 = V_1 \cup \{u\}$, and $U_2 = V_2 \setminus \{u\}$. This implies that $\phi' = (U_0, U_1, U_2)$ is an FScRDF on G. Now, $\omega_G^{FScR}(\phi') = \mid U_1 \mid +2 \mid U_2 \mid= (\mid V_1 \mid +1) + 2(\mid V_2 \mid -1) =\mid V_1 \mid +2 \mid V_2 \mid -1 < \omega_G^{FScR}(\phi) = \gamma_{FScR}(G)$. This is a contradiction. Hence, we obtain $V_2 = \emptyset$. As for the converse, we suppose that $V_2 = \emptyset$. Since ϕ is a γ_{FScR}-function on G, it simply follows that $V_0 = \emptyset$. Therefore, by Theorem 2, we conclude that $\gamma_{FScR}(G) = n$ and hence, (ii) is satisfied. This completes the proof. ■

Theorem 6. Let G be a graph and let $\phi = (V_0, V_1 = \emptyset, V_2)$ be an FScRDF on G. Then V_2 is a minimal fair secure dominating set on G if and only if for each $u \in V_2$, there exists a vertex $v \in V_0$ such that $N_G(v) \cap V_2 = \{u\}$ or $vw \notin E(G)$ for all $w \in (V_1 \cup V_2) \setminus \{u\}$.

Proof.

Let $\phi = (V_0, V_1, V_2)$ be an FScRDF on G. By Theorem 5(i), it follows that $V_1 \cup V_2$ is a fair secure dominating set on G.

($\Rightarrow$) Assume that $V_1 \cup V_2$ is a minimal fair secure dominating set on G. Then for every $u \in V_1 \cup V_2$, it implies that $(V_1 \cup V_2) \setminus \{u\}$ is not a fair secure dominating set on G. This follows that there exists $v \in V(G) \setminus ((V_1 \cup V_2) \setminus \{u\})$ such that $d_G(v, w) \neq 1$ for all $w \in (V_1 \cup V_2) \setminus \{u\}$. Suppose that $v \neq u$. It is worth noting that $V_1 \cup V_2$ is a fair secure dominating set on G; hence, v must be fair secure dominated by V_2, that is, there exists $x \in V_2$ such that $xv \in E(G)$. Consequently, this follows that $d_G(u, v) = 1$ and thus, $N_G(v) \cap V_2 = \{u\}$. On the other hand, suppose $v = u$. This means that $d_G(v, w) \neq 1$ for every $w \in (V_1 \cup V_2) \setminus \{u\}$.

($\Leftarrow$) Conversely, assume that for every $u \in V_2$, there exists $v \in V_0$ such that $N_G(v) \cap V_2 = \{u\}$. Then every $v \in V_0$ is not fair secure dominated by the set $(V_1 \cup V_2) \setminus \{u\}$. Suppose that for every $u \in V_2$, we have $d_G(u, w) \neq 1$ for any $w \in V_2 \setminus \{u\}$. This follows that $u$cannot be fair secure dominated by any vertex $w \in (V_1 \cup V_2) \setminus \{u\}$ and hence, $(V_1 \cup V_2) \setminus \{u\}$ is not a fair secure dominating set of G. Therefore, we conclude that $V_1 \cup V_2$ is a minimal fair secure dominating set on G.

This completes the proof. ■

Theorem 7. Let G be a graph and let $\phi = (V_0, V_1, V_2)$ be a γ_{FScR}-function on G. If $| V_2 | = 1$, then $\gamma_{FScR}(G) = n$.

Proof.

Let $\phi = (V_0, V_1, V_2)$ be a γ_{FScR}-function on G. Suppose $| V_2 | = 1$. Let $u \in V_2$. Assume for a moment that $\gamma_{FScR}(G) < n$. By Corollary 4, it follows that $| V_2 | < | V_0 |$. This implies that there exists $x, y \in V_0$ such that $d_G(x, u) = 1 = d_G(y, u)$ and $| N_G(x) \cap V_2 | = | N_G(y) \cap V_2 | \geq 1$. Consider the function $\phi^* = (V_0 \setminus \{y\}, V_1 \cup \{u, y\}, V_2 \setminus \{u\})$ on G. Then we have $V_0 \setminus \{y\} \neq \emptyset$ but $V_2 \setminus \{u\} = \emptyset$. This follows that ϕ^* is not an RDF on G. This is a contradiction. Therefore, we conclude that $\gamma_{FScR}(G) = n$. This completes the proof. ■

Corollary 8. Let G be a graph of order n. If $\gamma_{FScR}(G) < n$, then $| V_2 | \geq 2$.

Proof.

Let $\phi = (V_0, V_1, V_2)$ be a γ_{FScR}-function on G. Suppose $\gamma_{FScR}(G) < n$. Assume for a moment that $| V_2 | \leq 1$. Now, if $| V_2 | = 0$, then by Theorem 5, we obtain $| V_0 | = 0$. This follows that $\gamma_{FScR}(G) = \omega_G^{FScR}(\phi) = | V_1 | + 2 | V_2 | = | V_1 | = | V(G) | = n$. This is a contradiction. Moreover, we let $| V_2 | = 1$. Then by Theorem 7, we have $\gamma_{FScR}(G) = n$. This is again a contradiction to our assumption. Therefore, we conclude that $| V_2 | \geq 2$. This completes the proof. ■

The next theorem determines the bounds of a fair secure Roman domination number of any graph G.

Theorem 9. Let G be a graph of order n. Then the inequality holds:

$$max\,\{\gamma_R(G), \gamma_{FSc}(G)\} \leq \gamma_{FScR}(G) \leq min\,\{2\gamma_{FSc}(G), n\}.$$

Proof.

Let $\phi = (V_0, V_1, V_2)$ be a γ_{FScR}-function on G. Then by Theorem 5, it follows that $V_1 \cup V_2$ is a fair secure dominating set on G. Thus, we have $\gamma_{FSc}(G) \leq\mid V_1 \mid + \mid V_2 \mid \leq\mid V_1 \mid +2 \mid V_2 \mid = \omega_G^{FScR}(\phi) = \gamma_{FScR}(G)$. In addition, since every fair secure Roman dominating function is a Roman dominating function on G, it simply implies that $\gamma_R(G) \leq \gamma_{FScR}(G)$. Accordingly, we obtain $\max\{\gamma_{FSc}(G), \gamma_R(G)\} \leq \gamma_{FScR}(G)$. Now, let $V_0 = \emptyset$. Since ϕ is a γ_{FScR}-function on G, by Theorem 5, it means that $V_2 = \emptyset$. Hence, $\phi = (\emptyset, V_1 = V(G), \emptyset)$ is an FScRDF on G. In that case, we have $\gamma_{FScR}(G) \leq \omega_G^{FScR}(\phi) =\mid V_1 \mid +2 \mid V_2 \mid=\mid V_1 \mid=\mid V(G) \mid= n$. Hence, $\gamma_{FScR}(G) \leq n$. Let D be a γ_{FSc}-set on G, that is, $\gamma_{FSc}(G) =\mid D \mid$. Set $V_0' = V(G) \setminus D$, $V_1' = \emptyset$, and $V_2' = D$. Let $v \in V_0'$. Then there exists $u \in V_2'$ such that $uv \in E(G)$, $\phi^* = (V_0' \setminus \{v\}, \{v, u\}, V_2' \setminus \{u\})$ is an RDF on G, and for every $x, y \in V_0'$, $\mid N_G(x) \cap V_2' \mid=\mid N_G(y) \cap V_2' \mid> 0$. Hence, $\lambda = (V_0', V_1', V_2')$ is an FScRDF on G. Thus, we obtain $\gamma_{FScR}(G) \leq \omega_G^{FScR}(\lambda) =\mid V_1' \mid +2 \mid V_2' \mid= 2 \mid D \mid= 2\gamma_{FSc}(G)$. So, we have $\gamma_{FScR}(G) \leq 2\gamma_{FSc}(G)$. Accordingly, we obtain $\gamma_{FScR}(G) \leq \min\{n, 2\gamma_{FSc}(G)\}$. Therefore, we conclude that

$$\max\{\gamma_R(G), \gamma_{FSc}(G)\} \leq \gamma_{FScR}(G) \leq \min\{2\gamma_{FSc}(G), n\}.$$

This completes the proof. ■

Theorem 10. Let G be a non-complete graph and let $\phi = (V_0, V_1 = \emptyset, V_2)$ be an FScRDF on G such that $\mid N_G(x) \cap V_2 \mid=\mid N_G(y) \cap V_2 \mid\geq 2$ for any $x, y \in V_0$. Then ϕ is a γ_{FScR}-function on G if and only if V_2 is a γ_{FSc}-set on G. Moreover, $\gamma_{FScR}(G) = 2\gamma_{FSc}(G)$.

Proof.

Let $\phi = (V_0, V_1 = \emptyset, V_2)$ be an FScRDF on G.

($\Leftarrow$) Suppose V_2 is a γ_{FSc}-set on G. Then it follows that $V_0 = V(G) \setminus V_2$, and hence, $\phi^* = (V_0 \setminus \{v\}, V_1 = \{v, u\}, V_2 \setminus \{u\})$ is an RDF on G and for every $x, y \in V_0$, $\mid N_G(x) \cap V_2 \mid=\mid N_G(y) \cap V_2 \mid\geq 1$. Assume for a moment that ϕ is an FScRDF on G but not a γ_{FScR}-function on G. In that case, there exists a γ_{FScR}-function $\lambda = (W_0, W_1 = \emptyset, W_2)$ on G. Now, observe that $\gamma_{FScR}(G) = \omega_G^{FScR}(\lambda) =\mid W_1 \mid +2 \mid W_2 \mid= 2 \mid W_2 \mid< 2 \mid V_2 \mid=\mid V_1 \mid +2 \mid V_2 \mid= \omega_G^{FScR}(\phi)$. Hence, we get $\mid W_2 \mid<\mid V_2 \mid$. This is a contradiction since V_2 is a γ_{FSc}-set on G. Therefore, we conclude that ϕ is a γ_{FScR}-function on G.

($\Rightarrow$) Suppose that $\phi = (V_0, V_1 = \emptyset, V_2)$ is a γ_{FScR}-function on G. Then by Theorem 5, $V_1 \cup V_2 = V_2$ is a fair secure dominating set on G. Assume for a moment that V_2 is not a γ_{FSc}-set on G. Then there exists a γ_{FSc}-set on G, say, $V_2^{'}$. This follows that $V_2^{'} \subset V_2$. Define an RDF $f = (U_0, U_1, U_2)$ on G where $U_0 = V(G) \setminus V_2^{'}$, $U_1 = \emptyset$, and $U_2 = V_2^{'}$. Then f is an FScRDF on G, that is, for every $a \in U_0$, there exists $b \in U_2$ such that $ab \in E(G)$, $f^* = (U_0 \setminus \{a\}, \{a, b\}, U_2 \setminus \{b\})$ is an RDF on G, and for every $x, y \in U_0$, $| N_G(x) \cap U_2 | = | N_G(y) \cap U_2 | \geq 1$. Hence, $\omega_G^{FScR}(f) = 2 | U_2 | < 2 | V_2 | = \omega_G^{FScR}(\phi) = \gamma_{FScR}(G)$. This is a contradiction to the definition of a γ_{FScR}-function on G. Therefore, V_2 is a γ_{FSc}-set on G, that is, $\gamma_{FSc}(G) = | V_2 |$. Moreover, we have $\gamma_{FScR}(G) = \omega_G^{FScR}(\phi) = | V_1 | + 2 | V_2 | = 2 | V_2 | = 2\gamma_{FSc}(G)$. This completes the proof. ■

Theorem 11. Let G be a graph and let $\phi = (V_0, V_1, V_2)$ be an FScRDF on G. If $\gamma_{FScR}(G) < n$, then there exists $u \in V_2$ such that $| N_G(u) \cap V_0 | \geq 2$.

Proof.
Let $\phi = (V_0, V_1, V_2)$ be a γ_{FScR}-function on G. Suppose $\gamma_{FScR}(G) < n$. Then by Corollary 8, it implies that $| V_2 | \geq 2$. Assume for a moment that for all $u \in V_2$, $| N_G(u) \cap V_0 | = 1$. Since ϕ is a FScRDF on G, this implies that either for every $x \in V_0$, there exists $u, v \in V_2$ such that $d_G(u, x) = 1 = d_G(v, x)$ or by Theorem 2, for every $x, y \in V_0$, $| N_G(x) \cap V_2 | = | N_G(y) \cap V_2 | = 1$. If for every $x \in V_0$ there exists $u, v \in V_2$ such that $d_G(u, x) = 1 = d_G(v, x)$, then we obtain $| V_2 | > | V_0 |$. Then by Corollary 4, it follows that $\gamma_{FScR}(G) > n$. This is a contradiction to Theorem 9 that $\gamma_{FScR}(G) \leq n$. Now, if for every $x, y \in V_0$, $| N_G(x) \cap V_2 | = | N_G(y) \cap V_2 | = 1$, then it follows that $| V_2 | = | V_0 |$. Then by Theorem 2, we obtain $\gamma_{FScR}(G) = n$. This is a contradiction to our assumption that $\gamma_{FScR}(G) < n$. Consequently, we conclude that there exists $u \in V_2$ such that $| N_G(u) \cap V_0 | \geq 2$. This completes the proof. ■

Theorem 12. Let G be a graph with $| V(G) | = n$ and let $\phi = (V_0, V_1, V_2)$ be an FScRDF on G. If $| V_2 | = 2$ and $\gamma_{FScR}(G) < n$, then $| V_0 | \geq 3$ and for every $v \in V_0$, $| N_G(v) \cap V_2 | = 2$.

Proof.
Let $\phi = (V_0, V_1, V_2)$ be an FScRDF on G. Suppose $| V_2 | = 2$ and $\gamma_{FScR}(G) < n$. By Corollary 4, it follows that $| V_2 | < | V_0 |$. Hence, we have $| V_0 | \geq 3$. Assume for a moment that for any $v, v^{'} \in V_0$ such that $| N_G(v) \cap V_2 | = | N_G(v^{'}) \cap V_2 | \leq 1$. Now, if $| N_G(v) \cap V_2 | = | N_G(v^{'}) \cap V_2 | = 0$, then this is contrary to the definition of RDF on G. On the other hand, consider $| N_G(v) \cap V_2 | = | N_G(v^{'}) \cap V_2 | = 1$. Let $N_G(v) \cap V_2 = \{u\}$. Then $u \in V_2$. If $N_G(u) \cap V_0 = \{v\}$, then since $| V_2 | = 2$, there exists $w \in V_2 \setminus \{u\}$ such that $N_G(w) \cap V_0 = V_0 \setminus \{v\}$. Now, let $y \in V_0 \setminus \{v\}$ and consider $\phi^* = (V_0 \setminus \{y\}, V_1 \cup \{w, y\}, V_2 \setminus \{w\})$. Since $| V_2 | = 2$, $| V_0 | \geq 3$, and $N_G(u) \cap V_0 = \{v\}$, it implies that there exists $a \in$

$V_0 \setminus \{y, v\}$ such that $N_G(a) \cap (V_2 \setminus \{w, u\}) = \emptyset$. Hence, it means that ϕ^* is not an RDF on G, a contradiction. Therefore, we conclude that for every $v \in V_0$, $|N_G(v) \cap V_2| = 2$. This completes the proof. ■

Theorem 13. Let G be a graph with $|V(G)| = n$ and let $\phi = (V_0, V_1, V_2)$ be a γ_{FScR}-function on G for which $|V_2| \geq 2$ and $\gamma_{FScR}(G) < n$. If there exists $x \in V(G) \setminus V_2$ such that $|N_G(x) \cap V_2| = 1$ and for all $u \in V_2$, $|N_G(u) \cap V_0| \geq 2$, then $x \in V_1$.

Proof. Let $\phi = (V_0, V_1, V_2)$ be a γ_{FScR}-function on G for which $|V_2| \geq 2$ and $\gamma_{FScR}(G) < n$. Suppose there exists $x \in V(G) \setminus V_2$ such that $|N_G(x) \cap V_2| = 1$ and for all $u \in V_2$, $|N_G(u) \cap V_0| \geq 2$. Seeking a contradiction. Assume for a moment that $x \notin V_1$. Since $x \in V(G) \setminus V_2$, it means that $x \in V_0$. Since $\gamma_{FScR}(G) < n$, by Corollary 4, it follows that $2 \leq |V_2| < |V_0|$. Let $u \in N_G(x) \cap V_2$. Then $u \in V_2$. Since for all $u \in V_2$, $|N_G(u) \cap V_0| \geq 2$, there exists $y \in V_0 \setminus \{x\}$ such that $y \in N_G(u)$. Then consider the function $\phi^* = (V_0 \setminus \{y\}, V_1 \cup \{u, y\}, V_2 \setminus \{u\})$ on G. In that case, $N_G(x) \cap (V_2 \setminus \{u\}) = \emptyset$, implying that ϕ^* is not an RDF on G. A contradiction. Therefore, we conclude that $x \in V_1$. This completes the proof. ■

Theorem 14. Let G be a graph. Then the following hold: (i) $\gamma_{FScR}(G) = 1$ if and only if $G = K_1$; (ii) $\gamma_{FScR}(G) = 2$ if and only if $G \in \{K_2, \bar{K_2}\}$; and (iii) $\gamma_{FScR}(G) = 3$ if and only if $G \in \{K_3, \bar{K_3}, K_2 \cup K_1, P_3\}$.

Proof.
Let $\phi = (V_0, V_1, V_2)$ be a γ_{FScR}-function on G.

(i) Suppose $\gamma_{FScR}(G) = 1$. Seeking a contradiction. Assume for a moment that $G \neq K_1$. Then, we get $|V(G)| > 1$. Now, if $V_0 \neq \emptyset$, then by Theorem 5, we have $V_2 \neq \emptyset$. This implies that $\gamma_{FScR}(G) = \omega_G^{FScR}(\phi) = |V_1| + 2|V_2| > 1$, a contradiction. In addition, if $V_0 = \emptyset$, then $V_2 = \emptyset$ by Theorem 5. Thus, it follows that $\gamma_{FScR}(G) = \omega_G^{FScR}(\phi) = |V_1| + 2|V_2| = |V_1| = |V(G)| > 1$. This is again a contradiction. Therefore, we conclude that $G = K_1$. Conversely, let $G = K_1$. Then it follows that $V_1 = V(G)$. Thus, $\gamma_{FScR}(G) = \omega_G^{FScR}(\phi) = |V_1| = |V(G)| = 1$.

(ii) Assume that $\gamma_{FScR}(G) = 2$. Then it follows that $\omega_G^{FScR}(\phi) = |V_1| + 2|V_2| = 2$. Hence, we get $|V_2| \leq 1$. Suppose that $|V_2| = 1$. Then it implies that $|V_0| \geq 1$ and $|V_1| = 0$. Let $V_2 = \{v\}$. It follows that for all $w \in V_0$, $N_G(w) \cap V_2 = \{v\}$. Now, consider $\phi^* = (V_0 \setminus \{w\}, V_1 \cup \{v, w\}, V_2 \setminus \{v\})$. If $|V_0| \geq 2$, then $V_0 \setminus \{w\} \neq \emptyset$. However, since $|V_2| = 1$, it follows that $V_2 \setminus \{v\} = \emptyset$. Thus, ϕ^* is not an RDF on G, a contradiction. Hence, it means that $|V_0| = 1$ and so, $|V(G)| = |V_0| + |V_1| + |V_2| = 2$. Therefore, we obtain $G = K_2$. On the other hand, we suppose that $|V_2| = 0$. By Theorem 5, we have $|V_0| = 0$. And so, we obtain $\gamma_{FScR}(G) = \omega_G^{FScR}(\phi) = |V_1| + 2|V_2| = |V_1| = |V(G)| = 2$. Accordingly, we conclude that $G \in \{K_2, \bar{K_2}\}$. Conversely, let $G \in \{K_2, \bar{K_2}\}$.

Then it is clear that $\phi = (\emptyset, V_1 = V(G), \emptyset)$ is a γ_{FScR}-function on G. Thus, $\gamma_{FScR}(G) = \omega_G^{FScR}(\phi) = |V_1| = |V(G)| = 2$.

(iii) Assume that $\gamma_{FScR}(G) = 3$. Then $\omega_G^{FScR}(\phi) = |V_1| + 2|V_2| = 3$. This implies that $|V_2| \leq 1$. First, we consider $|V_2| = 0$. By Theorem 5, it follows that $|V_0| = 0$. Thus, we have $\gamma_{FScR}(G) = \omega_G^{FScR}(\phi) = |V_1| + 2|V_2| = |V_1| = |V(G)| = 3$. Hence, it means that $G \in \{K_3, \bar{K_3}, K_2 \cup K_1, P_3\}$. Secondly, we consider $|V_2| = 1$. Since $|V_1| + 2|V_2| = 3$, it follows that $|V_1| = 1$. Suppose $|V_0| \geq 2$ where V_0 is a fair vertex set on G. Let $v \in V_0$ and consider the function $\phi^* = (V_0 \setminus \{v\}, V_1 \cup \{u, v\}, V_2 \setminus \{u\})$ on G. In this case, $|V_0 \setminus \{v\}| \neq 0$, but $V_2 \setminus \{u\} = \emptyset$, implying that ϕ^* is not an RDF on G. This is a contradiction. Thus, it suffices to say that $|V_0| \leq 1$. Since $|V_2| = 1$ and ϕ is a γ_{FScR}-function on G, it follows that $|V_0| = 1$. Therefore, we conclude that $|V(G)| = |V_0| + |V_1| + |V_2| = 3$ and so, $G \in \{K_3, K_2 \cup K_1, P_3\}$. Conversely, let $G \in \{K_3, \bar{K_3}, K_2 \cup K_1, P_3\}$. Then it is easy to see that $\phi = (\emptyset, V_1 = V(G), \emptyset)$ is a γ_{FScR}-function on G. Thus, $\gamma_{FScR}(G) = \omega_G^{FScR}(\phi) = |V_1| = |V(G)| = 3$.

This completes the proof. ■

Theorem 15. Let G be a graph and let $\phi = (V_0, V_1 = \emptyset, V_2)$ be a γ_{FScR}-function on G. Then $\gamma_{FScR}(G) = \gamma_{FSc}(G) + 1$ if and only if $|V_2| = |V_0| = 1$.

Proof.

Let $\phi = (V_0, V_1 = \emptyset, V_2)$ be a γ_{FScR}-function on G.

($\Rightarrow$) Assume that $\gamma_{FScR}(G) = \gamma_{FSc}(G) + 1$. Then it follows that $\omega_G^{FScR}(\phi) = |V_1| + 2|V_2| = \gamma_{FSc}(G) + 1$. By Theorem 10, it implies that V_2 is a γ_{FSc}-set on G. Since $V_1 = \emptyset$ and $\gamma_{FSc}(G) = |V_2|$, it follows that $2|V_2| = |V_2| + 1$ and so, $|V_2| = 1$. Seeking a contradiction. Assume for a moment that $|V_0| \neq 1$. Then either $|V_0| = 0$ or $|V_0| \geq 2$. If $|V_0| = 0$, then it is a contradiction since $|V_2| = 1$ and ϕ is a γ_{FScR}-function on G. Now, consider $|V_0| \geq 2$ where V_0 is a fair vertex set on G. Let $x \in V_0$ and $u \in V_2$. Then consider a function $\phi^* = (V_0 \setminus \{x\}, V_1 \cup \{x, u\}, V_2 \setminus \{u\})$ on G. In this case, we get $|V_0 \setminus \{x\}| \neq 0$ since $|V_0| \geq 2$, but $V_2 \setminus \{u\} = \emptyset$ since $|V_2| = 1$. This implies that ϕ^* is not an RDF on G. A contradiction. Therefore, we conclude that $|V_2| = |V_0| = 1$.

($\Leftarrow$) Assume that $|V_2| = |V_0| = 1$. Since $V_1 = \emptyset$ and ϕ is a γ_{FScR}-function on G, by Theorem 10, V_2 is a γ_{FSc}-set on G, that is, $\gamma_{FSc}(G) = |V_2|$. Therefore, we end up with $\gamma_{FScR}(G) = \omega_G^{FScR}(\phi) = |V_1| + 2|V_2| = |V_2| + |V_2| = \gamma_{FSc}(G) + 1$.

This completes the proof. ■

Theorem 16. Let G be a graph. Then $\phi = (V_0, V_1 = \emptyset, V_2)$ is a γ_{FScR}-function on G and $\gamma_{FScR}(G) = \gamma_{FSc}(G) + 2$ if and only if there exist $x, y \in V(G)$ such that $|N_G(x) \cap N_G(y)| = n - \gamma_{FSc}(G)$.

Proof.
(⇒) Assume that $\phi = (V_0, V_1 = \emptyset, V_2)$ is a γ_{FScR}-function on G and $\gamma_{FScR}(G) = \gamma_{FSc}(G) + 2$. Then we have $\omega_G^{FScR}(\phi) = |V_1| + 2|V_2| = \gamma_{FSc}(G) + 2$. By Theorem 10, V_2 is a γ_{FSc}-set on G, that is, $\gamma_{FSc}(G) = |V_2|$. And since $V_1 = \emptyset$, it implies that $2|V_2| = |V_2| + 2$, and thus, we obtain $|V_2| = 2$. Let $V_2 = \{x, y\} \subseteq V(G)$. In that case, $V_0 = V(G) \setminus V_2$ is a fair vertex set on G, that is, for any $u, v \in V_0$, $|N_G(u) \cap V_2| = |N_G(v) \cap V_2| = 2$ and the function $\phi^* = (V_0 \setminus \{v\}, \{v, x\}, V_2 \setminus \{x\})$ is an RDF on G. Thus, it suffices to say that $|N_G(V_2)| = |V(G)| - |V_2|$. Since $V_2 = \{x, y\}$ and $\gamma_{FSc}(G) = |V_2|$, it follows that $|N_G(x) \cap N_G(y)| = |N_G(V_2)|$. Therefore, we conclude that $|N_G(x) \cap N_G(y)| = n - \gamma_{FSc}(G)$.
(⇐) Assume that there exist $x, y \in V(G)$ such that $|N_G(x) \cap N_G(y)| = n - \gamma_{FSc}(G)$. Then $\gamma_{FSc}(G) = |\{x, y\}|$. This follows that $\{x, y\}$ is a γ_{FSc}-set on G. Set $V_0 = V(G) \setminus \{x, y\}$, $V_1 = \emptyset$, and $V_2 = \{x, y\}$. Then V_2 is a γ_{FSc}-set on G. Since $V_1 = \emptyset$, by Theorem 10, it implies that $\phi = (V_0, V_1 = \emptyset, V_2)$ is a γ_{FScR}-function on G. To this end, we have $\gamma_{FScR}(G) = \omega_G^{FScR}(\phi) = |V_1| + 2|V_2| = |V_2| + |V_2| = \gamma_{FSc}(G) + 2$.
This completes the proof ■

Theorem 17. Let $G_j (j = 1, \dots, k)$ be the components of a graph G. Then $\phi = (V_0, V_1, V_2)$ is an FScRDF on G if and only if $\phi|_{G_j}$ is an FScRDF on G_j for each $j \in \{1, 2, \dots, k\}$. Hence,

$$\gamma_{FScR}(G) = \sum_{j=1}^{k} \gamma_{FScR}(G_j).$$

Proof.
Let G be a graph and let $G_1, \dots, G_k$ be the components of G.
(⇒) Assume that $\phi = (V_0, V_1, V_2)$ is an FScRDF on G. Then for each $j \in \{1, 2, \dots, k\}$, we let $V_0^j = V_0 \cap V(G_j)$, $V_1^j = V_1 \cap V(G_j)$, and $V_2^j = V_2 \cap V(G_j)$. Thus, we obtain $\phi|_{G_j} = (V_0^j, V_1^j, V_2^j)$ for each $j \in \{1, 2, \dots, k\}$. Let $v \in V_0^j$ for some $j \in \{1, 2, \dots, k\}$. This implies that $v \in V_0$. Since ϕ is an FScRDF on G, it follows that there exists $u \in V_2$ such that $uv \in E(G)$, $\phi^* = (V_0 \setminus \{v\}, V_1 \cup \{u, v\}, V_2 \setminus \{u\})$ is an RDF on G, and for any $x, y \in V_0$, $|N_G(x) \cap V_2| = |N_G(y) \cap V_2| \geq 1$. Since $d_G(x, y) = \infty$ for $x \in V(G_h)$ and $y \in V(G_l)$ where $h \neq l$, it follows that $u \in V_2^j$, $\left(\phi|_{G_j}\right)^* = (V_0^j \setminus \{v\}, V_1^j \cup \{u, v\}, V_2^j \setminus \{u\}\big)$ is an RDF on G_j, and for any $x, y \in V_0^j$, $|N_G(x) \cap V_2^j| = |N_G(y) \cap V_2^j| \geq 1$. Hence, it means that $\phi|_{G_j}$ is an FScRDF on G_j. If ϕ is a γ_{FScR}-function on G, then we obtain $\gamma_{FScR}(G) = |V_1| + 2|V_2| = \sum_{j=1}^{k} |V_1^j| + 2\sum_{j=1}^{k} |V_2^j| = \sum_{j=1}^{k}(|V_1^j| + 2|V_2^j|) \geq \sum_{j=1}^{k} \gamma_{FScR}(G_j)$.

($\Leftarrow$) Conversely, assume that $\phi\mid_{G_j} = (W_0^j, W_1^j, W_2^j)$ is an FScRDF on G_j for each $j \in \{1,2,\dots,k\}$. Then we set $V_0 = \bigcup_{j=1}^{k} W_0^j$, $V_1 = \bigcup_{j=1}^{k} W_1^j$, and $V_2 = \bigcup_{j=1}^{k} W_2^j$. This follows that $\phi = (V_0, V_1, V_2)$ is an RDF on G. Let $a \in V_0$. Then $a \in W_0^j$ for some $j \in \{1,2,\dots,k\}$. Since $\phi\mid_{G_j}$ is an FScRDF on G_j, there exists $b \in W_2^j$ such that $d_{G_j}(a,b) = 1$, $\left((\phi\mid_{G_j})^* = (W_0^j \setminus \{a\}, W_1^j \cup \{a,b\}, W_2^j \setminus \{b\}\right)$ is an RDF on G_j, and for any $x, y \in W_0^j$, $\mid N_G(x) \cap W_2^j \mid = \mid N_G(y) \cap W_2^j \mid \geq 1$. Since $W_i^j \subseteq V_i$ for each $i \in \{0,1,2\}$, it follows that $b \in V_2$, $\phi^* = (V_0 \setminus \{a\}, V_1 \cup \{a,b\}, V_2 \setminus \{b\})$ is an RDF on G, and for any $x, y \in V_0$, $\mid N_G(x) \cap V_2 \mid = \mid N_G(y) \cap V_2 \mid \geq 1$. Accordingly, we can say that ϕ is an FScRDF on G. Now, if $\phi\mid_{G_j}$ is a γ_{FScR}-function on G_j for all $j \in \{1,2,\dots,k\}$, then we have $\sum_{j=1}^{k} \gamma_{FScR}(G_j) = \sum_{j=1}^{k} (\mid W_1^j \mid + 2 \mid W_2^j \mid) = \sum_{j=1}^{k} \mid W_1^j \mid + 2\sum_{j=1}^{k} \mid W_2^j \mid = \mid V_1 \mid + 2 \mid V_2 \mid \geq \gamma_{FScR}(G)$. Therefore, we conclude that $\gamma_{FScR}(G) = \sum_{j=1}^{k} \gamma_{FScR}(G_j)$.

This completes the proof. ■

The next corollary is a direct consequence of Theorem 14(i) and Theorem 17.

Corollary 18. Let $G = \bar{K_n}$ where n is a positive integer. Then $\gamma_{FScR}(G) = n$.

The join of two graphs G and H is denoted by $G + H$ and is a graph with vertex set $V(G+H) = V(G) \cup V(H)$ and edge set $E(G+H) = E(G) \cup E(H) \cup \{uv : u \in V(G) \text{ and } v \in V(H)\}$.

Theorem 19. Let G be a connected graph and let $\phi = (V_0, V_1 = \emptyset, V_2)$ be an FScRDF on G. Then ϕ is a γ_{FScR}-function on G, $\mid V_2 \mid = 2$, and for every $u \in V_2$, $V_0 \subseteq N_G[u]$ if and only if $G = J + H$ where $J \in \{K_2, \bar{K_2}\}$ and H is any graph with $\mid V(H) \mid \geq 2$.

Proof.
Let $\phi = (V_0, V_1 = \emptyset, V_2)$ be an FScRDF on G.
($\Rightarrow$) Assume that ϕ is a γ_{FScR}-function on G, $\mid V_2 \mid = 2$, and for every $u \in V_2$, $V_0 \subseteq N_G[u]$. Then by Theorem 10, V_2 is a γ_{FSc}-set on G. Let $V_2 = \{x, y\}$. Then we can have $\langle V_2 \rangle = K_2$ (or $\langle V_2 \rangle = \bar{K_2}$). Since ϕ is a γ_{FScR}-function on G, it follows that $\mid V_0 \mid \geq 2$. In that case, we let $\langle V_0 \rangle = H$ where H is any graph with $\mid V(H) \mid = \mid V_0 \mid \geq 2$. Now, since for every $u \in V_2$, $V_0 \subseteq N_G[u]$ and V_2 is a γ_{FSc}-set on G, it implies that $G = K_2 + H$ (or $G = \bar{K_2} + H$). Hence, for every $v \in V(H) = V_0$, there exists $u \in V(K_2)$ (or $V(\bar{K_2})$) such that $uv \in E(G)$, $\phi^* = (V_0 \setminus \{v\}, V_1 = \{u, v\}, V_2 \setminus \{u\})$ is an RDF on G, and for any $a, b \in V_0$, $\mid N_G(a) \cap V_2 \mid = \mid N_G(b) \cap V_2 \mid = 2$. Therefore, we conclude that $G = J + H$ where $J \in \{K_2, \bar{K_2}\}$ and H is any graph with $\mid V(H) \mid \geq 2$.
($\Leftarrow$) Assume that $G = J + H$ where $J \in \{K_2, \bar{K_2}\}$ and H is any graph with $\mid V(H) \mid \geq 2$. Observe that for every $x, y \in V(H)$, $\mid N_G(x) \cap V(J) \mid = \mid N_G(y) \cap V(J) \mid > 0$. This implies that $V(H)$ is a fair vertex

set on G. In that case, we set $V_0 = V(H), V_1 = \emptyset$, and $V_2 \subseteq V(J)$. Suppose that $| V_2 |= 1$. Then let $z \in V_2 \subseteq V(J)$. If $J = \bar{K_2}$, then there exists $w \in J \setminus \{z\}$ such that $wz \notin E(G)$. Since $V_1 = \emptyset$, it follows that $\phi = (V_0, V_1 = \emptyset, V_2)$ is not an RDF on G. A contradiction. So, it follows that $V_2 \supseteq V(J)$ and thus, $V_2 = V(J)$. This implies that $| V_2 |=| V(J) |= 2$. Let $v \in V_0$. Then there exists $u \in V_2$ such that $uv \in E(G)$ and $\phi^* = (V_0 \setminus \{v\}, V_1 = \{u, v\}, V_2 \setminus \{u\})$ is an RDF on G. Hence, $\phi = (V_0 = V(H), V_1 = \emptyset, V_2 = V(J))$ is an FScRDF on G. In view of Theorem 16, $\phi = (V_0 = V(H), V_1 = \emptyset, V_2 = V(J))$ is a γ_{FScR}-function on G. Moreover, we concluded that for every $u \in V_2 = V(J), V(H) = V_0 \subseteq N_G[u]$.

This completes the proof. ■

The following results are immediate from Theorem 19.

Corollary 20. Let $G = K_2 + H$ or $G = \bar{K_2} + H$ where H is any graph with $| V(H) |\geq 2$. Then $\gamma_{FScR}(G) = 4$.

Remark 21. If $G = K_{m,n}$, where $m, n \geq 4$, then $\gamma_{FScR}(G) = 8$.

Remark 22. If $G = K_n$, where $n \geq 4$, then $\gamma_{FScR}(G) = 4$.

4. DISCUSSION

The results revealed that the existing Roman domination in graphs can be enhanced and strengthened, while providing a fair contribution of legions (vertex labeled 2) guarding the unsecured locations (vertex labeled 0). Some combinatorial properties were provided that describes the graph-theoretic configuration of the vertices (V_0, V_1, V_2) of a graph G labeled with 0, 1, and 2, which is similar to the results of existing papers in [16, 17, 18]. It is revealed that if $\gamma_{FScR}(G) < n$, then it is found out that $| V_1 |>| V_2 |\geq 2$. This result indicates that a whole graph to be secured needs more legions (weights) to defend the entire graph compared to the original Roman domination, in which the minimum weight is 2 with $| V_2 |= 1$. This implies that strengthening the defense of Roman domination requires more legions (more costly), which is consistent with the findings in [7].

Additionally, the study also dealt with the characterization of fair secure Roman domination number in small values, and it is found out that if $\gamma_{FScR}(G) \in \{1, 2, 3\}$, then $|V(G)| \in \{1, 2, 3\}$, and vice versa. This result is not parallel to the findings of the following papers: [2, 3, 4, 5, 16, 17, 18, 19, 20]. Moreover, combinatorial characterization of fair secure Roman dominating function in a graph with several components were provided, and it is consistent to the results in [16] and [17]. This means that a graph with several components, fair secure Roman dominating function can be observed in each independent component. Furthermore, since the minimum fair secure Roman domination number that satisfies $\gamma_{FScR}(G) < n$ where n is the order of a graph, is equal to 4, the study provided a characterization that if $\gamma_{FScR}(G) < n$, then it follows that $| V_2 |= 2$ and $| V_1 |= 0$ where a graph G is a form of a join of two graphs as stated in Theorem 19. This result leads to the notion that a fair secure Roman dominating function is realizable relative to Roman domination in graphs, which can be viewed similarly to the findings in [16] and [18].

5. CONCLUSION

This paper aims to introduce a new restricted version of Roman domination in graphs called fair secure Roman domination. Some important combinatorial properties were presented, and some bounds of the fair secure Roman domination number were given. Exact values of the fair secure Roman domination number of some classes of graphs were also determined. Moreover, necessary and sufficient conditions for a function to be fair secure Roman dominating in some graphs were obtained. This study highly recommends investigating and characterizing the fair secure Roman dominating function in graphs under some binary operations that include corona, Cartesian, and lexicographic products as future research.

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REFERENCES

[1] E. J. Cockayne, P. A. Dreyer Jr., S. M. Hedetniemi, and S. T. Hedetniemi, "Roman domination in graphs," Discrete Mathematics, vol. 278, no. 1–3, pp. 11–22, 2004.

[2] M. Chellali, T. W. Haynes, and S. T. Hedetniemi, "Roman and total domination," Quaestiones Mathematicae, vol. 38, no. 6, pp. 749–757, 2015.

[3] R. L. Pushpam and S. Padmapriea, "Restrained Roman domination in graphs," Transactions on Combinatorics, vol. 4, no. 1, pp. 1–17, 2015.

[4] H. A. Ahangar, M. A. Henning, V. Samodivkin, and I. G. Yero, "Total Roman domination in graphs," Applicable Analysis and Discrete Mathematics, vol. 10, no. 2, pp. 501–517, 2016.

[5] E. J. Cockayne, O. Favaron, and C. M. Mynhardt, "Secure domination, weak Roman domination and forbidden subgraphs," Bulletin of the Institute of Combinatorics and its Applications, vol. 39, pp. 87–100, 2003.

[6] Y. Caro, A. Hansberg, and M. Henning, "Fair domination in graphs," Discrete Mathematics, vol. 312, no. 19, pp. 2905–2914, 2012.

[7] E. L. Enriquez, "Fair secure domination in graphs," International Journal of Mathematics Trends and Technology, vol. 66, no. 2, pp. 49–57, 2020.

[8] J. A. Bondy and U. S. R. Murty, Graph Theory. London, U.K.: Springer, 2008.

[9] G. Chartrand, L. Lesniak, and P. Zhang, Graphs and Digraphs. Boca Raton, FL, USA: CRC Press, 2016.

[10] T. W. Haynes, S. Hedetniemi, and P. Slater, Fundamentals of Domination in Graphs. Boca Raton, FL, USA: CRC Press, 2013.

[11] B. S. Anand, J. A. Dayap, L. F. Casinillo, R. Pepper, and R. S. Nair, "On distance k-domination number of graphs under product operations," Gulf Journal of Mathematics, vol. 19, no. 1, pp. 117–124, 2025.

[12] L. F. Casinillo, "A note on Fibonacci and Lucas number of domination in path," Electronic Journal of Graph Theory and Applications, vol. 6, no. 2, pp. 317–325, 2018.

[13] L. F. Casinillo, "Odd and even repetition sequences of independent domination number," Notes on Number Theory and Discrete Mathematics, vol. 26, no. 1, pp. 8–20, 2020.
[14] L. F. Casinillo, "A closer look at a path domination number in grid graphs," Journal of Fundamental Mathematics and Applications, vol. 6, no. 1, pp. 18–26, 2023.
[15] E. J. Cockayne, R. M. Dawes, and S. T. Hedetniemi, "Total domination in graphs," Networks, vol. 10, no. 3, pp. 211–219, 1980.
[16] L. F. Casinillo "On interior Roman domination in graphs," Indonesian Journal of Combinatorics, vol. 9, no. 2, pp. 78-89, 2025.
[17] L. F. Casinillo, S. Canoy Jr. "Super hop Roman domination in graphs," European Journal of Pure and Applied Mathematics, vol. 18, no. 4, pp. 1-15, 2025.
[18] L. Casinillo, E. Casinillo "Outer-connected fair Roman dominating function in graphs," Annals of Mathematics and Computer Science, vol. 31, pp. 1-15, 2025.
[19] R. J. Fortosa, S. Canoy Jr. "Convex Roman domination in graphs," European Journal of Pure and Applied Mathematics, vol. 16, no. 3, pp. 1705-1716, 2023.
[20] A. A. Aradais, J. B. Cariaga, S. Canoy Jr. "Connected hop Roman domination in graphs," European Journal of Pure and Applied Mathematics, vol. 18, no. 1, pp. 1-13, 2025.