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Rainbow Connection Number of Octopus Iteration Graphs

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Abstract

The rainbow connection number of a graph G denoted by $rc(G)$ is the minimum number of colors used to color the edges in G, such that every pair of vertices is connected by a path with all different colors. In 2008, Chartrand, et al. first introduced the concept of rainbow connection numbers. They introduced it as an edge coloring on a graph that refers to the path of each pair of vertices. An octopus graph, with m legs denoted by O_m, is a graph constructed from a fan graph F_m and a star graph S_m. The graphs studied in this article are two classes of octopus iteration graphs, namely the octopus chain graph and the octopus ladder graph. The octopus chain graph, denoted by $O_2(n)$, is a graph constructed from n copies of O_2 and connecting one leg of the i-th copy to the $(i+1)-th$ copy, for every $i = 1, 2, \dots, n-1$. The octopus ladder graph, denoted by $O_2{}'(n)$, is a graph constructed from graph $O_2(n)$ by connecting one of vertex of degree two of the i-th copy to the $(i+1)-th$ copy. In this research, we determine the rainbow connection number of the octopus chain graphs $O_2(n)$ and The octopus ladder graphs $O_2{}'(n)$. We obtain that $rc(O_2(n)) = 3n$, for $n \geq 1$ and $rc(O_2{}'(n)) = 3n-1$, for $n \geq 2$.

Keywords: classes of octopus iteration graphs; octopus chain graph; octopus ladder graph; rainbow connection number.

Abstrak

Bilangan terhubung pelangi pada graf G dinotasikan dengan $rc(G)$ merupakan jumlah warna minimum yang digunakan untuk mewarnai sisi pada G, sehingga setiap pasang titik dihubungkan oleh suatu lintasan dengan warna yang berbeda semua. Pada tahun 2008, Chartrand dkk. pertama kali memperkenalkan konsep bilangan terhubung pelangi. Chartrand, dkk. memperkenalkannya sebagai pewarnaan sisi pada graf yang mengacu pada lintasan setiap pasang titiknya. Graf gurita dengan m kaki dinotasikan dengan O_m adalah graf yang dikonstruksi dari graf kipas F_m dan graf Bintang S_m. Graf yang dikaji dalam artikel ini merupakan dua kelas graf iterasi gurita, yaitu graf rantai gurita dan graf tangga gurita. Graf rantai gurita yang dinotasikan dengan $O_2(n)$ adalah graf yang dikonstruksi dari n copy graf O_2 dan menghubungkan satu kaki salinan ke$-i$ ke salinan ke$-i+1$, untuk setiap $i = 1,2, \dots, n-1$. Graf tangga gurita yang dinotasikan dengan $O_2{}'(n)$ adalah graf yang dibangun dari graf $O_2(n)$ dengan menghubungkan salah satu titik berderajat dua Salinan dari graph ke$-i$ ke salinan ke$-i+1$. Pada penelitian ini, ditentukan bilangan terhubung pelangi pada graf rantai gurita $O_2(n)$ dan graf tangga gurita $O_2{}'(n)$. Kami memperoleh bahwa $rc(O_2(n)) = 3n$ untuk $n \geq 1$ dan $rc(O_2{}'(n)) = 3n-1$, untuk $n \geq 2$.

Kata Kunci: *kelas graf iterasi gurita; graf rantai gurita; graf tangga gurita; bilangan terhubung pelangi.*

2020MSC: 05C15, 05C40.

1. INTRODUCTION

Graph theory is a branch of mathematics that studies the properties of graphs. In 1736, a scientist from Switzerland named Leonhard Euler attempted to solve the problem of the Königsberg bridge

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over the Pregel River in Russia. Leonhard Euler modeled the problem as a graph, referring to the land as vertex and the bridges as edge connecting them. A graph is a pair $G = (V, E)$ of sets such that $E \subseteq [V]^2$, thus the elements of E are 2-element subsets of V. The elements of V are the vertices of the graph G, and the elements of E are the edges of the graph G [1]. An example graph will be provided in Figure 1 to make it easier to understand. From Figure 1, graph G have a vertex-set $V = \{1,2,3,4\}$, the edge-set $E = \{(1,2); (2,3); (3,4); (1,4); (1,3)\}$, and the set of elements $[V]^2 = \{(1,2); (1,3); (1,4); (2,3); (2,4); (3,4)\}$ such that $E \subseteq [V]^2$.

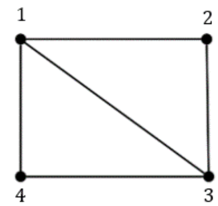

Figure 1. Graph G

Graph theory continues to evolve and produce new concepts that can be studied, one of which is the concept of rainbow connectivity. The concept of rainbow connectivity is an evolution of edge coloring, and this theory was first introduced in 2008 by Chartrand et al. The concept of the rainbow connection originated from communication issues between U.S. government agencies and their agents following the terrorist attacks on the eleventh of September 2001, which forced both parties to communicate through codes for national security reasons. Procedures must remain in place to ensure that agents have appropriate access to information. To address this issue, a transfer pathway between agents was created. However, due to the large number of agents and passwords, to prevent leaks, a minimum password requirement has been established so that every two agents have different passwords. The situation can be modeled by applying the concept of rainbow interconnectedness. In that problem, the nodes are represented as vertices and the edges as sides, with the minimum edges depicted as connected rainbow numbers in a graph.

Edge coloring is the assignment of colors to all edges of a graph. Another type of graph edge coloring is rainbow coloring, where no two edges in the graph have the same color [2]. Graph G is a rainbow path if each of its edges has a different color. A graph G is rainbow connected if it is connected by a rainbow path between every two vertices in G. The minimum number of colors used to color each edge of G such that G is rainbow connected is called the rainbow connection number of G, denoted as $rc(G)$. The distance $d_G(x, y)$ in G between two vertices x, y is the length of path from $x - y$ in G; if no such path is like that, we establishing $d(x, y) := \infty$. The farthest distance between two vertices in G is the diameter of G, denoted as $diam\ (G)$ [1]. A graph G with $diam\ (G) = 3$ is given in Figure 2.

Theorem 1 [2]. If G is a connected graph with $|V(G)| \geq 1$ and its diameter is $diam(G)$, then $diam(G) \leq rc(G)$.

The Figure 3 shows a graph G connected in a rainbow using 3 colors. Thus, $rc(G) = 3$. In this case, the $diam\ (G) = 3 = rc(G)$.

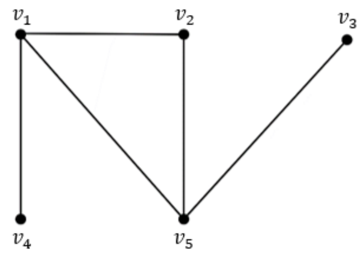

Figure 2. Graph G with $diam(G) = 3$

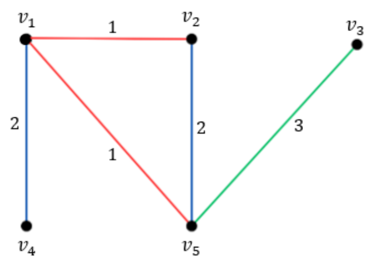

Figure 3. Rainbow coloring on graph G

A path graph is a non-empty graph $P = (V, E)$ from the set $V = \{x_0, x_1, \dots, x_k\}$ and $E = \{x_0x_1, x_1x_2, \dots, x_{k-1}x_k\}$, with all x_i being distinct. The vertices x_0 and x_k are connected by P where the number of edges in the path is referred to as the length. The length of the path with m vertices is denoted as P_m [1]. A fan graph F_m is a graph obtained by combining all the vertices from the path graph P_m into a vertex called the center point. Thus, the fan graph consists of $m + 1$ vertices and $2m - 1$ edges, where $m \geq 2$. A star graph is a graph with $m + 1$ vertices, with one vertex of degree m known as the center vertex connected to m other vertices of degree 1 called leaves, denoted as S_m. An octopus graph with m legs is denoted as O_m, $(m \geq 2)$ and is obtained by attaching the center vertex of the fan graph F_m, $(m \geq 2)$ to the center point of the star graph S_m, with any positive integer m. The octopus graph O_m has $2m + 1$ vertices and $3m - 1$ edges and has a diameter of $diam(O_m) = 2$. As an illustration, fan graph F_3, star graph S_6, and octopus graph O_3 are depicted in Figure 4.

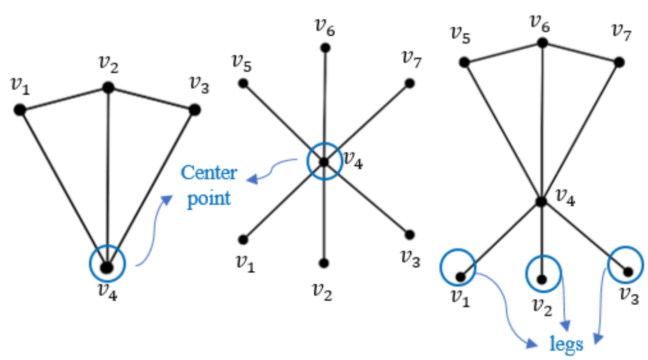

Figure 4. Fan Graph F_3**,** Star Graph S_6, and Octopus Graph O_3

In particular, the graph studied in this graph is constructed from the octopus graph O_2, as given in Figure 5. There are some previous research topics regarding rainbow-connected numbers such as the number of edge colorings [3], rainbow cycles in edge coloring graphs [4], antiprism graphs and complete graphs [5], flower "snark" graphs [6], on butterfly graphs, benes, and torus [7], on flower graphs and lemon graphs [8], on planter graphs and octopus graphs [9], on prism graphs and path graphs [10], on the amalgamation of tadpole graphs and sun graphs [11], as well as on the corona product of sandat graphs [12]. In addition to the rainbow connection number, the prime labeling of the octopus graph has been studied [13].

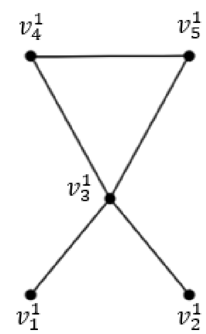

Figure 5. Octopus Graph O_2

Octopus iteration graphs are graphs formed by repeatedly duplicating the octopus graph and connecting the copies in a specific manner. In this paper, we determine the rainbow connection number of two classes of octopus iteration graphs: the octopus chain graph $O_2(n)$ and the octopus ladder graph $O_2{'}(n)$.

2. METHODS

This research employs literature study and analytical methods. The literature study method is a research approach that involves examining books on graph theory and research journals, particularly those studying the rainbow connection number of a graph and related topics. Meanwhile, the analytical method involves solving mathematical problems through mathematical proof and formulas. The steps used are as follows:

1. Conduct a literature study on rainbow connection number of graphs.
2. Define two classes of the octopus iteration graphs constructed from octopus graph O_2 namely the octopus chain graph $O_2(n)$, for every $n \geq 1$ and the octopus ladder graph $O_2{'}(n)$, for every $n \geq 2$.
3. Determine the rainbow connection number for the octopus chain graph $O_2(n)$ and octopus ladder graph $O_2{'}(n)$.
4. Formulate a theorem and prove its validity mathematically.
5. Make conclusions based on the results obtained.

3. RESULTS

In previous research, Fransiskus Fran et al. [9] determined the rainbow connection number of the octopus graph O_m, for $2 \leq m \leq 4$. In this section, we present definitions of octopus chain graphs and octopus ladder graphs. Then, we formulate two theorems about the rainbow connection number of these graphs and their proofs.

Definition 1. The octopus chain graph, denoted by $O_2(n)$, is a graph constructed from n copies of O_2 and connecting one leg of the i-th copy to the $(i+1)-th$ copy, for every $i = 1, 2, \dots, n-1$ and $n \geq 1$.

Graph $O_2(n)$ is given in Figure 6. Note that $O_2(1) \cong O_2$.

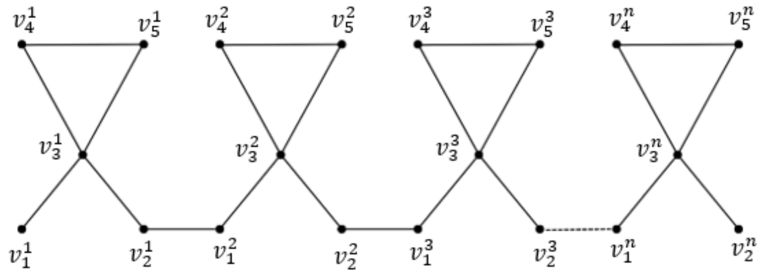

Figure 6. Graph $O_2(n)$

Definition 2. The octopus ladder graph denoted by $(O_2{}'(n))$ is a graph constructed from graph $O_2(n)$ by connecting vertex v_5^i to a vertex v_4^{i+1}, for every $i = 1, 2, \dots, n-1$.

Graph $O_2{}'(n)$ is given in Figure 7.

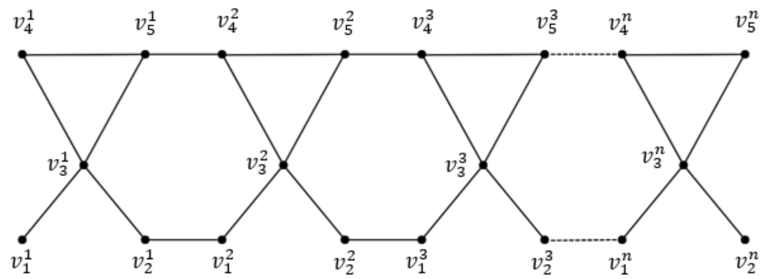

Figure 7. Graph $O_2{}'(n)$

The notation of the vertex v_j^i on the octopus chain graph $O_2(n)$ and the octopus chain graph $O_2(n)$ shows the vertex j on the i-th copy. Next, the following theorems give the rainbow connection number on octopus chain graphs and octopus ladder graphs.

Theorem 2. For any $n \geq 1$, $rc\big(O_2(n)\big) = 3n$.

Proof.
The octopus chain graph $O_2(n)$ is a graph that has a vertex-set:

$$V(O_2(n)) = \{v_j^i | 1 \leq i \leq n, j = 1,2,3,4,5\},$$

and edge-set:

$$E\big(O_2(n)\big) = \left\{v_1^i v_3^i, v_2^i v_3^i, v_4^i v_3^i, v_5^i v_3^i, v_4^i v_5^i \;\middle|\, 1 \leq i \leq n\right\} \cup \left\{v_2^i v_1^{i+1} \middle| 1 \leq i \leq n-1\right\},$$

then $\big|V\big(O_2(n)\big)\big| = 5n$ and $\big|E\big(O_2(n)\big)\big| = 6n - 1$, fo $n \geq 1$.

Next, we divide this proof into two cases.

Case 1. For $n = 1$.

First, we prove to a lower bound for $rc(O_2(1))$. Let c be any coloring of the edges of the graph $O_2(1)$. It will be shown that $rc(O_2(1)) \geq 3$. Consider that edges $v_1^1 v_3^1$ and $v_2^1 v_3^1$ must be colored differently. If not, there will be no rainbow path from v_1^1 to v_2^1 with a length at most 3. Assume $c(v_1^1 v_3^1) = 1$ and $c(v_2^1 v_3^1) = 2$. In addition, the remaining three edges must be colored differently otherwise there is no rainbow path from the leg to the vertex v_4^1 or v_5^1. So, it requires at least 3 colors to guarantee that there is a rainbow path from every pair of vertices in the graph $O_2(1)$. Thus, $rc(O_2(1)) \geq 3$.

Next, we prove the upper bound of $O_2(1)$.

Define the coloring of edges $c: E\big(O_2(1)\big) \to \{1,2,3\}$ as follows.

$$c(e) = \begin{cases} 1, & if\ e = v_1^1 v_3^1 \\ 2, & if\ e = v_2^1 v_3^1 \\ 3, & if\ e = v_3^1 v_4^1; v_3^1 v_5^1; v_4^1 v_5^1 \end{cases}$$

It will be shown that for each pair of vertices on $O_2(1)$, there is a rainbow path.

- For the pair of vertices v_1^1 to v_2^1,

The rainbow path is $v_1^1 - v_3^1 - v_2^1$.

- For the pair of vertices v_1^1 to v_4^1,

The rainbow path is $v_1^1 - v_3^1 - v_4^1$.

- For the pair of vertices v_1^1 to v_5^1,

The rainbow path is $v_1^1 - v_3^1 - v_5^1$.

- For the pair of vertices v_2^1 to v_5^1,

The rainbow path is $v_2^1 - v_3^1 - v_4^1 - v_5^1$.

For other pairs of vertices not explicitly mentioned in the cases above, the rainbow path for those pairs of vertices is contained in one of those cases. Since $rc(O_2(1)) \leq 3$ and $rc\big(O_2(1)\big) \geq 3$, then $rc(O_2(1)) = 3$.

Case 2. For $n \geq 2$.

First, we prove the lower bound for $rc(O_2(n))$, for $n \geq 2$. It will be shown that $rc(O_2(n)) \geq 3n$ Let c be any coloring of the edges of the graph $O_2(n)$. As in the proof of Case 1, the graph $O_2(1)$ (the first copy of $O_2(n)$, for $n \geq 2$) must be colored with 3 colors so that there is a rainbow path for every pair of vertices in the graph. Now, consider the edges $v_2^i v_1^{i+1}$, for $1 \leq i \leq n-1$ must be colored differently since the edges are bridges Furthermore, edges $v_1^i v_3^i$ and $v_2^i v_3^i$ for $2 \leq i \leq n$ must be colored differently such that there is a rainbow path with a length at most $3n$ from vertex v_1^1 to v_2^i. So, it means we requires at least $3 + (n-1) + 2(n-1) = 3n$ colors to guarantee that there is a rainbow path from every pair of vertices. Thus, $rc\big(O_2(n)\big) \geq 3n$. Then, we will determine the upper bound of $O_2(n)$, for $n \geq a$. Define the edge coloring $c: E\big(O_2(n)\big) \to \{1,2,\ldots,3n\}$ as follows.

$$c(e) = \begin{cases} 1, & if\ e = v_1^1 v_3^1 \\ 2, & if\ e = v_2^1 v_3^1, \\ 3, & if\ e = v_3^1 v_4^1; v_3^1 v_5^1; v_4^1 v_5^1 \end{cases}$$

and for every $2 \leq i \leq n$,

$$c(e) = \begin{cases} 1, & if\ e = v_3^i v_4^i \\ 3, & if\ e = v_3^i v_5^i \\ 3i-2, & if\ e = v_2^{i-1} v_1^i\,. \\ 3i-1, & if\ e = v_1^i v_3^i \\ 3i, & if\ e = v_2^i v_3^i; v_4^i v_5^i \end{cases}$$

It will be shown that for each pair of vertices on $O_2(n)$, there is a rainbow path.

- For the pair of vertices v_1^2 to v_2^n, the rainbow path is $v_1^2 - v_3^2 - v_2^2 - v_1^3 - v_3^3 - v_2^3 - \cdots - v_2^n$.
- For the pair of vertices v_1^2 to v_4^n, the rainbow path is $v_1^2 - v_3^2 - v_2^2 - v_1^3 - v_3^3 - v_5^3 - v_4^3 - \cdots - v_4^n$.
- For the pair of vertices v_4^2 to v_2^n, the rainbow path is $v_4^2 - v_3^2 - v_2^2 - v_1^3 - v_3^3 - v_2^3 - \cdots - v_2^n$.
- For the pair of vertices v_4^2 to v_5^n, the rainbow path is $v_4^2 - v_3^2 - v_2^2 - v_1^3 - v_3^3 - v_4^3 - v_5^3 - \cdots - v_5^n$.
- For the pair of vertices v_5^2 to v_2^n, the rainbow path is $v_5^2 - v_3^2 - v_2^2 - v_1^3 - v_3^3 - v_2^3 - \cdots - v_2^n$.
- For the pair of vertices v_5^2 to v_5^n, the rainbow path is $v_5^2 - v_3^2 - v_2^2 - v_1^3 - v_3^3 - v_4^3 - v_5^3 - \cdots - v_5^n$.

For other pairs of vertices not explicitly mentioned in the cases above, the rainbow path for those pairs of vertices is contained in one of those cases. Since $rc(O_2(n)) \leq 3n$ and $rc(O_2(n)) \geq 3n$, then $rc(O_2(n)) = 3n$. ∎

The coloring of graph $O_2(n)$ can be seen in Figure 7.

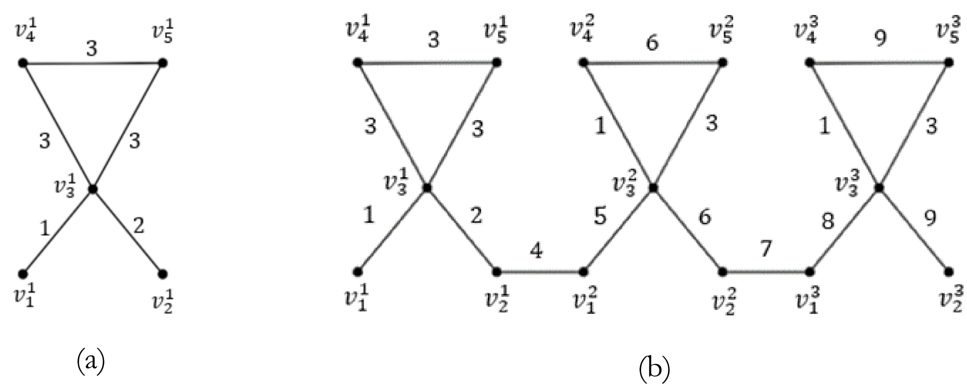

Figure 7. (a) Graph $O_2(1)$; **(b)** Graph $O_2(3)$.

Theorem 3. For any $n \geq 1$,

$$rc(O_2{}'(n)) = \begin{cases} 3, & for\ n = 1, \\ 3n-1, & for\ n \geq 2. \end{cases}$$

Proof.
Graph $O_2{}'(n)$ has a vertex-set:

$$V(O_2'(n)) = \{v_j^i | 1 \leq i \leq n, j = 1,2,3,4,5\},$$

and edge-set

$$E\big(O_2'(n)\big) = \big\{v_1^i v_3^i, v_2^i v_3^i, v_4^i v_3^i, v_5^i v_3^i, v_4^i v_5^i \,\big|1 \leq i \leq n\big\} \cup \big\{v_2^i v_1^{i+1}, v_5^i v_4^{i+1} \big| 1 \leq i \leq n-1\big\}.$$

Then the number of vertices is $\left|V\big(O_2'(n)\big)\right| = 5n$ and the number of edges is $\left|E\big(O_2'(n)\big)\right| = 7n-2$. It is clear that $O_2'(1) \cong O_2(1) \cong O_2$ and $rc(O_2'(1)) = rc(O_2(1)) = 2$. So, we now consider for $n \geq 2$. We divided it into two cases.

Case 1. For $n = 2$.

First, we prove the lower bound for $rc(O_2'(2))$. Since the $diam\ (O_2'(2)) = 5$, based on Theorem 1 in section 1 we have $rc\big(O_2'(2)\big) \geq 5$. Next, we will determine the upper bound for $O_2'(2)$. Let c be any edge coloring of the graph $O_2'(2)$.

Define the edge coloring $c: E\big(O_2'(2)\big) \to \{1,2,3,4,5\}$ as follows.

$$c(e) = \begin{cases} 1, & if\ e = v_1^1 v_3^1 \\ 2, & if\ e = v_2^1 v_3^1, \\ 3, & if\ e = v_3^1 v_4^1; v_3^1 v_5^1; v_4^1 v_5^1 \end{cases}$$

and

$$c(e) = \begin{cases} 2, & if\ e = v_1^2 v_3^2; v_4^2 v_5^2 \\ 3, & if\ e = v_3^2 v_4^2 \\ 4, & if\ e = v_2^1 v_1^2; v_5^1 v_4^2 \\ 5, & if\ e = v_2^2 v_3^2; v_3^2 v_5^2 \end{cases}.$$

It will be shown that for each pair of vertices on $O_2'(2)$, there is a rainbow path.

- For the pair of vertices v_1^1 to v_2^2, the rainbow path is $v_1^1 - v_3^1 - v_2^1 - v_1^2 - v_3^2 - v_2^2$.
- For the pair of vertices v_1^1 to v_4^2, the rainbow path is $v_1^1 - v_3^1 - v_5^1 - v_4^2$.
- For the pair of vertices v_1^1 to v_5^2, the rainbow path is $v_1^1 - v_3^1 - v_2^1 - v_1^2 - v_3^2 - v_5^2$.
- For the pair of vertices v_2^1 to v_4^2, the rainbow path is $v_2^1 - v_1^2 - v_3^2 - v_4^2$.
- For the pair of vertices v_4^1 to v_1^2, the rainbow path is $v_4^1 - v_3^1 - v_2^1 - v_1^2$.
- For the pair of vertices v_4^1 to v_2^2, the rainbow path is $v_4^1 - v_5^1 - v_4^2 - v_3^2 - v_2^2$.
- For the pair of vertices v_4^1 to v_5^2, the rainbow path is $v_4^1 - v_5^1 - v_4^2 - v_5^2$.
- For the pair of vertices v_5^1 to v_1^2, the rainbow path is $v_5^1 - v_4^2 - v_3^2 - v_1^2$.

For other pairs of vertices not explicitly mentioned in the cases above, the rainbow path for those pairs of vertices is contained one of those cases. Because $rc\left(O_2'^{(2)}\right) \leq 5$ and $rc\big(O_2'(2)\big) \geq 5$, it can be concluded that $rc\left(O_2'^{(2)}\right) = 5$.

Case 2. For $n \geq 3$.

First, we will prove the lower bound for $rc(O_2{}'(n))$. Since the $diam(O_2{}'(n)) = 3n-1$ based on Theorem 1.1 in section 1 we have $rc\big(O_2{}'(n)\big) \geq 3n-1$. Next, we will determine the upper bound for $O_2{}'(n)$. Let c be any edge coloring of the graph $O_2{}'(n)$.

Define the edge coloring $c: E\big(O_2{}'(n)\big) \to \{1,2,\ldots,3n-1\}$ as follows.

$$c(e) = \begin{cases} 1, & if\ e = v_1^1 v_3^1 \\ 2, & if\ e = v_2^1 v_3^1, v_1^2 v_3^2, v_4^2 v_5^2, \\ 3, & if\ e = v_3^1 v_4^1, v_3^1 v_5^1, v_4^1 v_5^1, v_3^2 v_4^2 \end{cases}$$

and

$$c(e) = \begin{cases} 4, & if\ e = v_2^1 v_1^2; v_5^1 v_4^2 \\ 5, & if\ e = v_2^2 v_3^2; v_3^2 v_5^2 \end{cases},$$

and for every $3 \leq i \leq n$,

$$c(e) = \begin{cases} 3i-4, & if\ e = v_3^i v_4^i \\ 3i-3, & if\ e = v_2^{i-1} v_1^i, v_5^{i-1} v_4^i \\ 3i-2, & if\ e = v_1^i v_3^i; v_4^i v_5^i \\ 3i-1, & if\ e = v_2^i v_3^i; v_3^i v_5^i \end{cases}.$$

It will be shown that for each pair of vertices on $O_2{}'(n)$, there is a rainbow path.

- For the pair of vertices v_1^3 to v_2^n, the rainbow path is $v_1^3 - v_3^3 - v_2^3 - v_1^4 - v_3^4 - v_2^4 - \cdots - v_2^n$.
- For the pair of vertices v_1^3 to v_4^n, the rainbow path is $v_1^3 - v_3^3 - v_5^3 - v_4^4 - \cdots - v_4^n$.
- For the pair of vertices v_1^3 to v_5^n, the rainbow path is $v_1^3 - v_3^3 - v_2^3 - v_1^4 - v_3^4 - v_5^4 - \cdots - v_5^n$.
- For the pair of vertices v_2^3 to v_4^n, the rainbow path is $v_2^3 - v_1^4 - v_3^4 - v_4^4 - \cdots - v_4^n$.
- For the pair of vertices v_4^3 to v_1^n, the rainbow path is $v_4^3 - v_3^3 - v_2^3 - v_1^4 - \cdots - v_1^n$.
- For the pair of vertices v_4^3 to v_2^n, the rainbow path is $v_4^3 - v_5^3 - v_4^4 - v_3^4 - v_2^4 - \cdots - v_2^n$.
- For the pair of vertices v_4^3 to v_5^n, the rainbow path is $v_4^3 - v_5^3 - v_4^4 - v_5^4 - \cdots - v_5^n$.
- For the pair of vertices v_5^3 to v_1^n, the rainbow path is $v_5^3 - v_4^4 - v_3^4 - v_1^4 - \cdots - v_1^n$.

For other pairs of vertices not explicitly mentioned in the cases above, the rainbow path for those pairs of vertices is contained in one of those cases. Since $rc\big(O_2{}'(n)\big) \leq 3n-1$ and $rc\big(O_2{}'(n)\big) \geq 3n-1$, it can be concluded that $rc\big(O_2{}'(n)\big) = 3n-1$. ■

The coloring of graph $O_2{}'(n)$ can be seen in Figure 8.

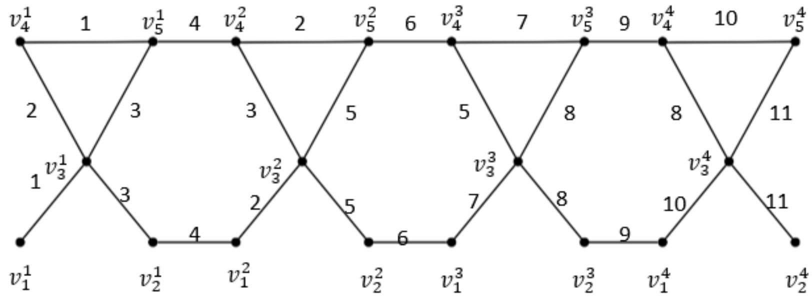

Figure 8. Graph $O_2'(4)$.

4. DISCUSSIONS

In this research, we determined the rainbow connection number for two classes of octopus iteration graphs: the octopus chain graph $O_2(n)$ for $n \geq 1$ and the octopus ladder graph $O_2^{\prime(n)}$ for $n \geq 2$. The results indicate that the rainbow connection number for the octopus chain graph is $3n$, where n , which represent the number of iterations (copies) of the octopus graphs O_2. This means that to achieve a rainbow connection in the octopus chain graph, at least $3n$ distinct colors are required to color the edges. For the octopus ladder graph, the rainbow connection number is slightly different. It is $3n-1$ for $n=1$ and $3n$ for $n \geq 2$. This suggests that the octopus ladder graph requires fewer colors to achieve a rainbow connection than the octopus chain graph, for $n \geq 2$. Based on the Definition 1 and Definition 2, it is known that $diam(O_2(n)) = diam\big(O_2'(n)\big) = 3n-1$, for $n \geq 2$. The rainbow connection number of the octopus ladder graph is more than its diameter. This case also occurs in several classes of graphs, including cycle graphs C_n for odd $n \geq 5$, wheel graphs W_n for $n \geq 7$ [2] and fan graphs F_n for $n \geq 7$ [14], planter graphs R_n for even $n \geq 2$ and the octopus graph O_2 [9], sandat graph $St(n)$, for $n \geq 3$ [15]. Unlike the octopus chain graph, the octopus ladder graph has a rainbow connection number equal to its diameter. This case also occurs in several classes of graphs, including complete graphs K_n, cycle graphs C_n for odd $n \geq 4$, and wheel graphs W_n for $4 \leq n \leq 6$ [2], fan graphs F_n for $3 \leq n \leq 6$ [14], origami graphs O_n, for $n \geq 3$ [16], and triangular snake graphs T_n, for $n \geq 2$ [17]. From the results of this research and several previous researches, it can be seen that graphs with the same diameter and similar structure (even in the same graph class) can have different rainbow connection numbers.

5. CONCLUSSION

In the results and discussion above, it can be concluded that the rainbow connection number of octopus chain graph $O_2(n)$ for $n \geq 1$, while the rainbow connection number in the octopus ladder graph $O_2'(n)$ for $n \geq 2$ is equal to the diameter. The results for the rainbow connection number of the octopus chain graph $O_2(n)$ and octopus ladder graph $O_2'(n)$ are as follows:

1. Rainbow connection number of the octopus chain graphs $(O_2(n))$

$$rc\big(O_2(n)\big) = 3n, \text{ for } n \geq 1.$$

2. Rainbow connection number of octopus ladder graphs $(O_2'(n))$

$$rc\big(O_2'(n)\big) = \begin{cases} 3, & for\ n = 1 \\ 3n-1, & for\ n \geq 2 \end{cases}.$$

In this research, we study rainbow connection number of two classes of octopus iteration graphs, namely the octopus chain graph and the octopus ladder graph constructed from an octopus graph O_2. Determination of rainbow connection number of graphs constructed from the octopus graphs O_m for $m \geq 3$ still an open problem.

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