InPrime: Indonesian Journal of Pure and Applied Mathematics
Vol. 6, No. 2 (2024), pp. 135 – 147, doi: 10.15408/inprime.v6i2.41126
p-ISSN 2686-5335, e-ISSN: 2716-2478

On Performance Measures of a Fuzzy Priority Queue in a Transient Regime using the L-R Method

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Abstract

In this paper, we proposed the performance measures of a Markovian queue with non-preemptive priority under transient conditions in a fuzzy environment. We study the transient performance measures of an FM/FM/1 queue with absolute priority. We analyze the characteristics of this model in a fuzzy environment by the L-R method of a fuzzy number $\tilde{A} = \langle m, a, b \rangle_{LR}$, where m, a, b are exponential functions of time—considering arrivals and services as triangular fuzzy numbers. The L-R method turns out to be very short and more practical compared to other methods, such as the DSW method (Dong, Shah and Wong), the α-cuts method, or the centroid method, because it allows us to obtain the membership functions, modal values, and supports of different performance measures and facilitates the graphical representation of the results obtained. A numerical example illustrates the validity of the method and the results obtained using it.

Keywords: performance measures; non-preemptive priority; triangular fuzzy numbers; LR method.

Abstrak

Pada artikel ini, kami mengusulkan ukuran kinerja antrian Markovian dengan prioritas non-preemptive dalam kondisi transien di lingkungan fuzzy. Kami mempelajari ukuran kinerja transien dari antrian FM/FM/1 dengan prioritas mutlak. Kami menganalisis karakteristik model ini dalam lingkungan fuzzy *dengan metode L-R dari bilangan fuzzy* $\tilde{A} = \langle m, a, b \rangle_{LR}$, *dengan* m, a, b *adalah fungsi eksponensial terhadap waktu dengan mempertimbangkan kedatangan dan pelayanan sebagai bilangan fuzzy segitiga. Metode L-R ternyata sangat singkat dan lebih praktis dibandingkan dengan metode lain, seperti: metode DSW (Dong, Shah dan Wong), metode a-cuts, atau metode centroid, karena metode ini memungkinkan kita untuk mendapatkan fungsi keanggotaan, nilai modus, dan* supports *dari ukuran kinerja yang berbeda dan memfasilitasi representasi grafis dari hasil yang diperoleh. Contoh numerik menggambarkan validitas dari metode dan hasil yang diperoleh dengan menggunakan metode yang kami usulkan.*

Kata Kunci: *ukuran kinerja; prioritas non-preemptive; bilangan fuzzy segitiga; metode* LR.

2020MSC: 60K25, 03E72.

1. INTRODUCTION

Studies on queues date back to the beginning of the 20th century with the work of A. K. Erlang [1], followed by substantial contributions from Prade (1980), Chapman and Kolmogorov (Wiki. Org), while Zadeh and Bellman [2] introduced the concept of fuzziness according to which imprecise information could be processed in decision-making problems. The Fuzzy set theory is often used

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Submitted August 29th, 2024, Revised October 24th, 2024,
Accepted for publication October 28th, 2024, Published Online November 5th, 2024

when imprecise and uncertain information needs to be modelled. Both concepts have evolved and have been the subject of several scientific publications. This paper on the performance measures of the fuzzy priority FM/FM/1 model under transient conditions completes a non-exhaustive list of publications on priority and/or non-priority fuzzy queues. These include: Kalpana and Anusheela [3], perform a fuzzy priority analysis with two classes of customers with unequal service rates using a mixed integer nonlinear programming approach; Shanmugasundaram and Vankatesh [4], studied the priority queuing model in a fuzzy environment. They optimized a fuzzy priority queuing model in which the arrival, service, and retry rates are fuzzy numbers.

The DSW algorithm determines the membership functions for the performance measures. Pardo and de la Fuente [5], provided a more realistic description of priority discipline queuing models using fuzzy set theory. They developed and optimized two fuzzy queuing models with priority discipline. Ritha and Yasodai 2021 [6][7], published a fuzzy queueing model with Erlang service distribution. Both the arrival rate and service rate were triangular fuzzy numbers using a simple, robust ranking algorithm to transform the fuzzy arrival rate and fuzzy service rate into accurate numbers. All these known results are published in steady state. Very few are published in a transient state. We can cite the work of J. Alonge et al. [8], which uses the α-flexible cut method to compute Markovian fuzzy performance parameters of the FM/FM/1 system in a transient state.

In view of this rich and abundant existing literature, there needs to be more analysis relative to performance measures in transient regimes with non-preemptive priority. This supports the fundamental question in preparing this paper: "Is it possible to evaluate the performance measures of non-preemptive customers in transient conditions in a fuzzy environment? To this question, we formulate the following hypothesis: if we fuzzyfy the results already available, Lama and Mabela [9], using fuzzy arithmetic with the L-R method, we can calculate the performance parameters of this category of customer in a fuzzy environment.

The originality of this reflection lies in the fact that no publication has been made in the existing literature on transient performance measurements with absolute priority in a fuzzy environment. In this paper, we limit ourselves to the computations of the following parameters: the numbers and times of priority and non-priority customers in the system and the queue at time $t \geq 0$ in a transient regime based on the imprecision of the arrivals and services of the two categories of customers. To achieve this, our paper is organized as follows: first, a reminder of L-R model Fuzzy Sets, then a fuzzy analysis of the performance measures of the FM/FM/1 system, and end with a numerical example applying the L-R method. We plan to conduct a similar study on the FM/FM/C and FM/G/1 models.

2. METHODS

Definition 1. Considering a frame of reference E, according to (Zadeh [10]) and (Kaufmann [11]), a fuzzy subset A of this frame of reference E is characterized by a membership function μ of E in the interval of real numbers [0,1]. The application formally defines a fuzzy subset μ, but to bring us back to the language of classical mathematics, we will speak of a fuzzy set A denoted $\tilde{A}$, and denote $\mu_{\tilde{A}}$ its membership function.

Definition 2. (Ziane [12]) A fuzzy set $\tilde{A}$ defined on the set of real numbers ℝ is said to be a fuzzy number if its membership function $\mu_{\tilde{A}}: \mathbb{R} \rightarrow [0,1]$ has the following characteristics:

1. $\mu_{\tilde{A}}$ is convex, i.e. $\mu_{\tilde{A}}(\lambda a + (1-\lambda)b) \geq \min(\mu_{\tilde{A}}(a), \mu_{\tilde{A}}(b))$, $\lambda \in [0,1]$.
2. $\mu_{\tilde{A}}$ is normal, i.e., there is $x \in \mathbb{R}$ such that $\mu_{\tilde{A}}(x) = 1$
3. $\tilde{A}$ is superior and semi-continuous.
4. $supp(\tilde{A})$ is bounded in $\mathbb{R}$

Definitions 3. (Kalpana and Anusheela [3]) For a fuzzy number $\tilde{A}$ of a reference frame E we give the following definitions:

$$\text{Core } N(\tilde{A}) = \{x \in E \ \backslash \ \mu_{\tilde{A}}(x) = 1\}, \tag{1}$$

$$\text{Support } \boldsymbol{S}(\tilde{A}) = \{x \in E \ \backslash \ \mu_{\tilde{A}}(x) \neq 0\}, \tag{2}$$

$$\text{Height } \boldsymbol{H}(\tilde{A}) = max\{\mu_{\tilde{A}}(x) \backslash x \in E\}. \tag{3}$$

Definition 4. (Banerjee and Roy, [13]) The alpha-cut (α) of a fuzzy set $\tilde{A}$ of reference frame E, is a subset denoted $\tilde{A}_\alpha$ containing the elements of E such that their membership degree is greater than or equal to α

$$\tilde{A}_\alpha = \{x \in E \colon \mu_{\tilde{A}}(x) \geq \alpha, \ \alpha \in [0,1]\}. \tag{4}$$

Definition 5. (Banerjee and Roy [13], Kalpana and Anusheela [3]) A fuzzy set $\tilde{A}$ is normal if and only if $\boldsymbol{H}(\tilde{A}) = 1$. A fuzzy set $\tilde{A}$ is convex if and only if N($\tilde{A}$)$\geq \min\{\ \mu_{\tilde{A}}(x),\ \mu_{\tilde{A}}(y)\} \ \forall x, y \in \tilde{A} \ \forall \alpha \in [0,1]$.

Definition 6. A fuzzy number $\tilde{A}$ is said to be positive if and only if $\forall x < 0, \mu_{\tilde{A}}(x) = 0$ and negative if and only if $\forall x > 0, \mu_{\tilde{A}}(x) = 0$.

Definition 7. A fuzzy number $\tilde{A}$ is said to be triangular if it is represented by $\tilde{A}(a, b, c,)$ such that its membership function is given by :

$$\mu_{\tilde{A}}(x) = \begin{cases} \frac{x-a}{b-a} \ ; a < x \leq b, \\ \frac{c-x}{c-b} \ ; b < x \leq c, \\ 0; \quad elsewhere, \end{cases} \tag{5}$$

where b is the modal value of $\tilde{A}$.

Definition 8. (Dubois and Prade [14]) A fuzzy number $\tilde{A}$ is said to be L-R if and only if there exist three numbers $m, a > 0, b > 0$, and two real-valued positive and decreasing continuous functions L and R on $[0,1]$ such that :

$$L(0) = R(0) = 1, \tag{6}$$

$$R(1) = 0, R(x) > 0, \quad \lim_{\infty} R(x) = 0, \tag{7}$$

$$\mu_{\tilde{A}}(x) = \begin{cases} L\left(\frac{m-x}{a}\right) ; if\ x \in [m-a, m], \\ R\left(\frac{x-m}{b}\right) ; if\ x \in [m, m+b], \\ 0 ; \qquad elsewhere. \end{cases} \tag{8}$$

The L-R representation of a fuzzy number $\tilde{A}$ is $\tilde{A} = \langle m, a, b \rangle_{LR}$, m is called mean value or modal value, a and b are called left span or spread and right span or spread of $\tilde{A}$ respectively.

$$supp(\tilde{A}) = [m-a, m+b].$$

L-R fuzzy number arithmetic [12] [13].

Let $\tilde{A} = \langle m, a, b \rangle_{LR}\ and\ \tilde{B} = \langle n, c, d \rangle_{LR}$ be two L-R fuzzy numbers, then we have:

$$\tilde{A} + \tilde{B} = \langle m, a, b \rangle_{LR} + \langle n, c, d \rangle_{LR} = \langle m+n, a+c, b+d \rangle_{LR}, \tag{9}$$

$$\tilde{A} - \tilde{B} = \langle m, a, b \rangle_{LR} - \langle n, c, d \rangle_{LR} = \langle m-n, a+d, b+c \rangle_{LR}. \tag{10}$$

If $\tilde{A}$ and $\tilde{B}$ are positives (Mukeba [15]), (Ziane [12]), then :

$$\tilde{A}.\tilde{B} = \langle m, a, b \rangle_{LR}.\langle n, c, d \rangle_{LR} = \langle m.n, mc + n\,a - ac, md + nb + bd \rangle_{LR}. \tag{11}$$

The quotient of two L-R fuzzy numbers is approximately given by (Mukeba et al. [16])

$$\frac{\tilde{A}}{\tilde{B}} = \frac{\langle m,a,b \rangle_{LR}}{\langle n,c,d \rangle_{LR}} \cong \langle \frac{m}{n}, \frac{(m-a)d + a(n+d)}{n(n+d)}, \frac{(m+b)c + b(n-c)}{n(n-c)} \rangle_{LR}. \tag{12}$$

In the transient regime, the variables m, a and b are functions of time.

2. RESULTS

In this paper, performance parameters or measures are limited to the average number of customers and the average waiting time of priority and non-priority customers in the system and the queue in a fuzzy environment.

2.1. Priority customers

This category of customers acts in the system as if it were a standard FIFO discipline queue. It is assumed that these customers arrive in the system following a Poisson process of rate λ_1, and their services follow the exponential law of rate μ and service intensity $\rho = \frac{\lambda_1}{\mu}$ during a time interval t in a standard transient regime. We have: (Lama and Mabela [9], Alonge et al. [8]).

$$\bar{N}_{s1}(t) = \frac{\rho_1\left(1-e^{-(\mu-\lambda_1)t}\right)}{1-\rho_1} = \frac{\lambda_1(1-e^{\alpha})}{\mu-\lambda_1}, \text{with } \alpha = -(\mu-\lambda)t \text{ and } \rho_1 = \frac{\lambda_1}{\mu}. \tag{13}$$

$$\bar{T}_{s1}(t) = \frac{\rho_1(1-e^{\alpha})}{1-\rho_1(\lambda_1+(\mu-\lambda_1)e^{\alpha}} = \frac{\lambda_1(1-e^{\alpha})}{(\mu-\lambda_1)(\lambda_1+(\mu-\lambda_1)e^{\alpha}}\ . \tag{14}$$

$$\bar{N}_{q1}(t) = \frac{\rho_1^2-(\rho_1^2-\rho_1+1)e^{\alpha}}{1-\rho_1} = \frac{\lambda_1^2-(\lambda_1^2-\lambda_1+\mu)e^{\alpha}}{\mu-\lambda_1}. \tag{15}$$

$$\bar{T}_{q1}(t) = \frac{\rho_1^2-(\rho_1^2-\rho_1+1)e^{\alpha}}{(1-\rho_1)\,(\lambda_1+(\mu-\lambda_1)e^{\alpha}} = \frac{\lambda_1^2-(\lambda_1^2-\lambda_1+\mu)e^{\alpha}}{(\mu-\lambda_1)(\lambda_1+(\mu-\lambda_1)e^{\alpha}}\,. \tag{16}$$

2.2. Non-priority customers (see Lama and Mabela [9])

For this customer category, the task is not easy because the arrivals and service times of priority customers in the system heavily disrupt their services. In this case of absolute priority, the service of a non-priority customer can only result in the total absence of a priority customer in the system. As with priority customers, non-priority customers arrive in the system at respective rates λ_1 *and* λ_2 so that the average arrival rate is $\lambda = \lambda_1 + \lambda_2$.

The theorems that give the results below have been stated and demonstrated:

The average service rate for both customer categories is μ.

The traffic intensity in the system is $\rho = \frac{\lambda}{\mu} = \frac{\lambda_1}{\mu} + \frac{\lambda_2}{\mu} = \rho_1 + \rho_2$.

The performance parameters for non-priority customers over a period t are :

$$\bar{N}_{s_2}(t) = \frac{\rho_2}{1-\rho}\left(1+\frac{\lambda_2\rho_1}{1-\rho_1}\right)(1-e^{\alpha}) = \frac{\lambda_2}{\mu-\lambda}\left(\frac{(\mu-\lambda_1)+\lambda_2\lambda_1}{\mu-\lambda_1}\right)(1-e^{\alpha}), \tag{17}$$

$$\bar{T}_{s_2}(t) = \frac{\rho_2}{1-\rho}\left(1+\frac{\lambda_2\rho_1}{1-\rho_1}\right)\frac{(1-e^{\alpha})}{\lambda+(\mu-\lambda)e^{\alpha}} = \frac{\lambda_2}{\mu-\lambda}\left(\frac{(\mu-\lambda)+\lambda_2\lambda_1}{\mu-\lambda_1}\right)\frac{(1-e^{\alpha})}{\lambda+(\mu-\lambda)e^{\alpha}}, \tag{18}$$

$$\bar{N}_{q_2}(t) = \frac{\rho_2}{1-\rho}\left(1+\frac{\lambda_2\rho_1}{1-\rho_1}\right)\frac{(1-e^{\alpha})\lambda}{\lambda+(\mu-\lambda)e^{\alpha}}\lambda = \frac{\lambda_2}{\mu-\lambda}\left(\frac{\mu-\lambda+\lambda_2\lambda_1}{\mu-\lambda_1}\right)\frac{(1-e^{\alpha})}{\lambda+(\mu-\lambda)e^{\alpha}}\lambda, \tag{19}$$

$$\bar{T}_{q_2}(t) = \frac{\rho_2}{1-\rho}\left(1+\frac{\lambda_2\rho_1}{1-\rho_1}\right)\frac{(1-e^{\alpha})}{\lambda+(\mu-\lambda)e^{\alpha}} - \frac{\rho}{\lambda} = \frac{\lambda_2}{\mu-\lambda}\left(\frac{(\mu-\lambda)+\lambda_2\lambda_1}{\mu-\lambda_1}\right)\frac{(1-e^{\alpha})}{\lambda+(\mu-\lambda)e^{\alpha}} - \frac{1}{\mu}. \tag{20}$$

Now, suppose that all arrival and service rates are fuzzy numbers of type L-R denoted by $\tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{\mu}$; then, under these conditions, the M/M/ 1 model becomes a Markovian fuzzy model FM/FM/1 where FM represents a fuzzified exponential distribution.

Theorem 1. In the fuzzy domain, the expressions for the average numbers and average times of non-priority customers in the system are exponential functions $\tilde{\lambda}$ and $\tilde{\mu}$.

Proof.

The proof of this theorem has been established in the standard transient regime, so it suffices to fuzzyfy the parameters λ and μ to obtain the results.

Hence the previous expressions (17) - (20) of the performance measures become :

$$\tilde{N}_{s1}(t) = \frac{\tilde{\rho}_1\left(1-e^{-(\tilde{\mu}-\widetilde{\lambda 1})\,t}\right)}{1-\tilde{\rho}_1} = \frac{\tilde{\lambda}_1(1-e^{\tilde{\alpha}})}{\tilde{\mu}-\tilde{\lambda}_1}, \text{ with } \tilde{\alpha} = -\left(\tilde{\mu}-\widetilde{\lambda 1}\right)t \text{ and } \tilde{\rho}_1 = \frac{\tilde{\lambda}_1}{\tilde{\mu}}, \tag{21}$$

$$\tilde{T}_{s1}(t) = \frac{\tilde{\lambda}_1(1-e^{\tilde{\alpha}})}{(\tilde{\mu}-\tilde{\lambda}_1)(\tilde{\lambda}_1+(\tilde{\mu}-\tilde{\lambda}_1)e^{\tilde{\alpha}}}, \tag{22}$$

$$\tilde{N}_{q1}(t) = \frac{\tilde{\lambda}_1^2-(\tilde{\lambda}_1^2-\tilde{\lambda}_1+\tilde{\mu}^2)e^{\tilde{\alpha}}}{\tilde{\mu}^2-\tilde{\mu}\tilde{\lambda}_1}, \quad (23)$$

$$\tilde{T}_{q1}(t) = \frac{\tilde{\lambda}_1^2-(\tilde{\lambda}_1^2-\tilde{\lambda}+\tilde{\mu})e^{\tilde{\alpha}}}{(\tilde{\mu}-\tilde{\lambda}_1)(\tilde{\lambda}_1+(\tilde{\mu}-\tilde{\lambda}_1)e^{\tilde{\alpha}})}, \quad (24)$$

$$\tilde{N}_{s_2}(t) = \frac{\tilde{\lambda}_2}{\tilde{\mu}-\tilde{\lambda}}\left(\frac{\tilde{\mu}-\tilde{\lambda}_1+\tilde{\lambda}_2\tilde{\lambda}_1}{\tilde{\mu}-\tilde{\lambda}_1}\right)\left(1-e^{\tilde{\alpha}}\right), \quad (25)$$

$$\tilde{T}_{s_2}(t) = \frac{\tilde{\lambda}_2}{\tilde{\mu}-\tilde{\lambda}}\left(1+\frac{\tilde{\lambda}_2\tilde{\lambda}_1}{\tilde{\mu}-\tilde{\lambda}_1}\right)\frac{(1-e^{\tilde{\alpha}})}{\tilde{\lambda}+(\tilde{\mu}-\tilde{\lambda})e^{\tilde{\alpha}}}, \quad (26)$$

$$\tilde{N}_{q_2}(t) = \frac{\tilde{\lambda}_2}{\tilde{\mu}-\tilde{\lambda}}\left(1+\frac{\tilde{\lambda}_2\tilde{\lambda}_1}{\tilde{\mu}-\tilde{\lambda}_1}\right)\frac{(1-e^{\tilde{\alpha}})}{\tilde{\lambda}+(\tilde{\mu}-\tilde{\lambda})e^{\alpha}}.\tilde{\lambda}, \quad (27)$$

$$\tilde{T}_{q_2}(t) = \frac{\tilde{\lambda}_2}{\tilde{\mu}-\tilde{\lambda}}\left(1+\frac{\tilde{\lambda}_2\tilde{\lambda}_1}{\tilde{\mu}-\tilde{\lambda}_1}\right)\frac{(1-e^{\tilde{\alpha}})}{\tilde{\lambda}+(\tilde{\mu}-\tilde{\lambda})e^{\alpha}} - \frac{1}{\tilde{\mu}}. \quad (28)$$

As $\tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{\mu}$ are L-R fuzzy numbers, we use L-R fuzzy arithmetic. A fuzzy queue is stable if and only if $\tilde{\lambda} < \tilde{\mu}$, in other words, $\sup\{\text{supp}(\tilde{\lambda})\} < \inf\{supp(\tilde{\mu})\}$.

2.3. Numerical Example

The Markovian arrivals of non-priority customers at the counter of a commercial bank in Kinshasa are given by the following triangular fuzzy numbers: $\tilde{\lambda}_1 = [2,3,4]$, $\tilde{\lambda}_2 = [1,2,3]$ and $\lambda = [3,5,7]$ following the Poisson distribution. Their services are also Markovian with the exponential law of triangular fuzzy rate $\tilde{\mu} = [8,9,10]$.

Questions:

1. Calculate the performance parameters for the two types of customer.
2. Graph these results.

The degrees of membership of these fuzzy numbers are such that :

$$u_{\tilde{\lambda}_1} = \begin{cases} \frac{x-2}{1} & 2 \leq x < 3, \\ \frac{4-x}{1} & 3 \leq x < 4, \\ 0 & elsewhere. \end{cases} \qquad u_{\tilde{\lambda}_2} = \begin{cases} \frac{x-1}{1} & 1 \leq x < 2, \\ \frac{3-x}{1} & 2 \leq x < 3, \\ 0 & elsewhere. \end{cases} \qquad u_{\tilde{\lambda}} = \begin{cases} \frac{x-3}{2} & 3 \leq x < 5, \\ \frac{7-x}{2} & 5 \leq x < 7, \\ 0 & elsewhere. \end{cases}$$

$$u_{\tilde{\mu}} = \begin{cases} \dfrac{x-8}{1} & 8 \leq x < 9, \\ \dfrac{10-x}{1} & 9 \leq x < 10, \\ 0 & elsewhere. \end{cases}$$

The L-R forms of $\tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{\lambda}, \tilde{u}$ are as follows:

$$\tilde{\lambda}_1 = \langle 3,1,1 \rangle_{LR}, \quad \tilde{\lambda}_2 = \langle 2,1,1 \rangle_{LR}, \quad \tilde{\lambda} = \langle 5,2,2 \rangle_{LR}, \quad \tilde{u} = \langle 9,1,1 \rangle_{LR}.$$

Let $x = e^{-(\tilde{\mu}-\tilde{\lambda})t}$.

1. Parameter calculations

1.1. Average number of priority customers in the system during a time t :

$$\tilde{N}_{s1}(t) = \frac{\tilde{\rho}_1\left(1-e^{-(\tilde{\mu}-\tilde{\lambda}_1)\,t}\right)}{1-\tilde{\rho}_1} = \frac{\tilde{\lambda}_1(1-x)}{\tilde{\mu}-\tilde{\lambda}_1}, \text{ with } x = e^{-(\tilde{\mu}-\tilde{\lambda}_1)t},$$

$$\tilde{N}_{s1}(t) = \frac{\langle 3,1,1\rangle_{LR}(1-x)}{\langle 9,1,1\rangle_{LR}-\langle 3,1,1\rangle_{LR}} = \frac{\langle 3,1,1\rangle_{LR}-\langle 3x,x,x\rangle_{LR}}{\langle 9,1,1\rangle_{LR}-\langle 3,1,1\rangle_{LR}} = \frac{\langle 3-3x,1+x,1+x\rangle_{LR}}{\langle 6,2,2\rangle_{LR}}$$

$$= \langle \frac{1-x}{2}, \frac{1}{4}, \frac{1}{2} \rangle_{LR},$$

$$x = e^{-\langle 9,1,1\rangle_{LR}-\langle 3,1,1\rangle_{LR}\,t} = e^{-\langle 6,2,2\rangle_{LR}\,t},$$

$$supp\left(\tilde{N}_{s1}(t)\right) = \left]\frac{1-x}{2}-\frac{1}{4}, \frac{1-x}{2}+\frac{1}{2}\right[= \left]\frac{1-2e^{-\langle 6,2,2\rangle_{LR}\,t}}{4}, \frac{2-e^{-\langle 6,2,2\rangle_{LR}\,t}}{2}\right[,$$

$$mod\left(\tilde{N}_{s1}(t)\right) = \frac{1}{2}-\frac{e^{-6t}}{2}.$$

1.2. Average waiting time for a priority customer in the system over a period of time t :

$$\tilde{T}_{s1}(t) = \frac{\tilde{\lambda}_1(1-x)}{(\tilde{\mu}-\tilde{\lambda}_1)(\tilde{\lambda}_1+(\tilde{\mu}-\tilde{\lambda}_1)x)} = \frac{\langle 3,1,1\rangle_{LR}(1-x)}{(\langle 9,1,1\rangle_{LR}-\langle 3,1,1\rangle_{LR})(\langle 3,1,1\rangle_{LR}+(\langle 9,1,1\rangle_{LR}-\langle 3,1,1\rangle_{LR})x)}$$

$$= \frac{\langle 3-3x,1+x,1+x\rangle_{LR}}{\langle 6,2,2\rangle_{LR}(\langle 3,1,1\rangle_{LR}+\langle 6x,2x,2x\rangle_{LR})} = \frac{\langle 3-3x,1+x,1+x\rangle_{LR}}{\langle 6,2,2\rangle_{LR}\times\langle 3+6x,1+2x,1+2x\rangle_{LR}}$$

$$= \frac{\langle 3-3x,1+x,1+x\rangle_{LR}}{\langle 18+36x,10+20x,14+28x\rangle_{LR}} = \langle \frac{3-3x}{18+36x}, \frac{60+96x-48x^2}{(18+36x)(32+64x)}, \frac{48+94x-24x^2}{(18+36x)(8+16x)} \rangle_{LR},$$

$$supp\left(\tilde{T}_{s1}(t)\right) = \left]\frac{3-3x}{18+36x}-\frac{60+96x-48x^2}{(18+36x)(32+64x)}, \frac{3-3x}{18+36x}+\frac{48+94x-24x^2}{(18+36x)(8+16x)}\right[$$

$$= \left]\frac{36-144x^2}{(18+36x)(32+64x)}, \frac{72+108x-88x^2}{(18+36x)(8+16x)}\right[= \left]\frac{1-2x}{16+32x}, \frac{2-x}{4+8x}\right[= \left]\frac{1-2e^{-\langle 6,2,2\rangle_{LR}\,t}}{16+32x}, \frac{2-e^{-\langle 6,2,2\rangle_{LR}\,t}}{4+8x}\right[,$$

$$mod\left(\tilde{T}_{s1}(t)\right) = \frac{1-e^{-6t}}{6+12e^{-6t}}.$$

1.3. Average number of priority customers in the queue over time t:

$$\tilde{N}_{q1}(t) = \frac{\tilde{\lambda}_1^2-(\tilde{\lambda}_1^2-\tilde{\lambda}_1+\tilde{\mu}^2)x}{\tilde{\mu}^2-\tilde{\mu}\tilde{\lambda}_1} = \frac{\langle 3,1,1\rangle_{LR}{}^2-(\langle 3,1,1\rangle_{LR}{}^2-\langle 3,1,1\rangle_{LR}+\langle 9,1,1\rangle_{LR}{}^2)\,x}{\langle 9,1,1\rangle_{LR}{}^2+\langle 9,1,1\rangle_{LR}.\langle 3,1,1\rangle_{LR}}$$

$$= \frac{\langle 9,5,7\rangle_{LR}-\langle 87x,23x,27x\rangle_{LR}}{\langle 54,28,32\rangle_{LR}} = \frac{\langle 9-87x,5+23x,7+27x\rangle_{LR}}{\langle 54,28,32\rangle_{LR}}$$

$$= \langle \frac{9-87x}{54}, \frac{558-154x}{4644}, \frac{630-978x}{1456} \rangle_{LR},$$

$$supp\left(\tilde{N}_{q1}(t)\right) = \left]\frac{9-87x}{54}-\frac{558-154x}{4644}, \frac{9-87x}{54}+\frac{630-978x}{1456}\right[= \left]\frac{2-55x}{43}, \frac{8-30x}{13}\right[$$

$$= \left]\frac{2-55e^{-\langle 6,2,2\rangle_{LR}t}}{43}, \frac{8-30e^{-\langle 6,2,2\rangle_{LR}t}}{13}\right[,$$

$$mod\left(\widetilde{N}_{q1}(t)\right) = \frac{3-29e^{-6t}}{18} = 0.16 - 1.6e^{-6t}.$$

1.4. The average time for priority customers in the queue at a given moment t:

$$\widetilde{T}_{q1}(t) = \frac{\widetilde{\lambda}_1^2-(\widetilde{\lambda}_1^2-\widetilde{\lambda}+\widetilde{\mu}^2)e^{\widetilde{\alpha}}}{(\widetilde{\mu}^2-\widetilde{\mu}\widetilde{\lambda}_1)(\widetilde{\lambda}_1+(\widetilde{\mu}-\widetilde{\lambda}_1)e^{\widetilde{\alpha}})} = \frac{\langle 9-87x,5+23x,7+27x\rangle_{LR}}{\langle 54,28,32\rangle_{LR}(\langle 3,1,1\rangle_{LR}+\langle 6x,2x,2x\rangle_{LR})}$$

$$= \frac{\langle 9-87x,5+23x,7+27x\rangle_{LR}}{\langle 162+324x,110+220x,182+364x\rangle_{LR}}$$

$$= \langle\frac{9-87x}{162+324x},\frac{2448-7212x-24216x^2}{(162+324x)(344+688x)},\frac{2124-948x-10392x^2}{(162+324x)(52+104x)}\rangle_{LR},$$

$$supp\left(\widetilde{T}_{q1}(t)\right) =]\frac{9-87x}{162+324x} - \frac{2448-7212x-24216x^2}{(162+324x)(344+688x)},\frac{9-87x}{162+324x} + \frac{2124-948x-10392x^2}{(162+324x)(52+104x)}[$$

$$=]\frac{648-16524x-35640x^2}{(162+324x)(344+688x)},\frac{2592-4536x-19440x^2}{(162+324x)(344+688x)}[=]\frac{2-55e^{-\langle 6,2,2\rangle_{LR}t}}{167+344e^{-\langle 6,2,2\rangle_{LR}t}},\frac{12-45e^{-\langle 6,2,2\rangle_{LR}t}}{13+26e^{-\langle 6,2,2\rangle_{LR}t}}[,$$

$$mod\left(\widetilde{T}_{q1}(t)\right) = \frac{3-29e^{-6t}}{54+108e^{-6t}}.$$

Average number and average waiting time of non-priority customers in the system :

$$Let\ x = e^{-(\widetilde{\mu}-\widetilde{\lambda})t} = e^{-(\langle 9,1,1\rangle_{LR}-\langle 5,2,2\rangle_{LR})t} = e^{-\langle 4,3,3\rangle_{LR}t}.$$

1.5. The average number of customers is not given priority for a period of time.

$$\widetilde{N}_{s_2}(t) = \frac{\widetilde{\lambda}_2}{\widetilde{\mu}-\widetilde{\lambda}}\left(\frac{\widetilde{\mu}-\widetilde{\lambda}_1+\widetilde{\lambda}_2\widetilde{\lambda}_1}{\widetilde{\mu}-\widetilde{\lambda}_1}\right)\left(1-e^{-(\widetilde{\mu}-\widetilde{\lambda})t}\right) = \frac{\langle 2,1,1\rangle_{LR}}{\langle 4,3,3\rangle_{LR}}\left(\frac{\langle 6,2,2\rangle_{LR}+\langle 6,4,6\rangle_{LR}}{\langle 6,2,2\rangle_{LR}}\right)(1-x)$$

$$= \frac{\langle 2,1,1\rangle_{LR}.\langle 12,6,8\rangle_{LR}}{\langle 4,3,3\rangle_{LR}.\langle 6,2,2\rangle_{LR}}(1-x) = \langle 1,\frac{14}{15},\frac{25}{8}\rangle_{LR}(1-x)$$

$$= \langle 1,\frac{14}{15},\frac{25}{8}\rangle_{LR} - \langle x,\frac{14x}{15},\frac{25x}{8}\rangle_{LR} = \langle 1-x,\frac{14(1+x)}{15},\frac{25(1+x)}{8}\rangle_{LR},$$

$$supp\left(\widetilde{N}_{s_2}(t)\right) =]1-x-\frac{14(1+x)}{15},1-x+\frac{25(1+x)}{8}[$$

$$=]\frac{1-29e^{-\langle 4,3,3\rangle_{LR}t}}{15},\frac{23+24e^{-\langle 4,3,3\rangle_{LR}t}}{8}[.$$

$$mod\left(\widetilde{N}_{s_2}(t)\right) = 1-e^{-4t}.$$

1.6. The average waiting time in the system for non-priority customers during a period of t.

$$\widetilde{T}_{s_2}(t) = \frac{\widetilde{\lambda}_2}{\widetilde{\mu}-\widetilde{\lambda}}\left(\frac{\widetilde{\mu}-\widetilde{\lambda}_1+\widetilde{\lambda}_2\widetilde{\lambda}_1}{\widetilde{\mu}-\widetilde{\lambda}_1}\right)\frac{(1-x)}{\widetilde{\lambda}+(\widetilde{\mu}-\widetilde{\lambda})x}$$

$$= \frac{\langle 2,1,1\rangle_{LR}}{\langle 9,1,1\rangle_{LR}-\langle 5,2,2\rangle_{LR}}\left(\frac{\langle 9,1,1\rangle_{LR}-\langle 3,1,1\rangle_{LR}+\langle 3,1,1\rangle_{LR}.\langle 2,1,1\rangle_{LR}}{\langle 3,1,1\rangle_{LR}.\langle 2,1,1\rangle_{LR}}\right)\frac{(1-x)}{\langle 5,2,2\rangle_{LR}+(\langle 9,1,1\rangle_{LR}-\langle 5,2,2\rangle_{LR})x}$$

$$= \frac{\langle 24-24x,18+x,36+x\rangle_{LR}}{\langle 102+96x,108+92x,272+360x\rangle_{LR}} = \langle\frac{1-x}{5+4x},\frac{362+165x-268x^2}{(5+4x)(392+456x)},\frac{288+133x-88x^2}{(5+4x)(12+4x)}\rangle_{LR}.$$

$$supp\left(\tilde{T}_{s_2}(t)\right) =]\frac{1-x}{5+4x} - \frac{362+165x-268x^2}{(5+4x)(392+456x)}, \frac{1-x}{5+4x} + \frac{288+133x-88x^2}{(5+4x)(12+4x)}]$$

$$=]\frac{30-101x-188x^2}{(5+4x)(392+456x)}, \frac{300+125x-92x^2}{(5+4x)(12+4x)}[\ =]\frac{10-47e^{-\langle 4,3,3\rangle_{LR}t}}{392+456x}, \frac{60-23e^{-\langle 4,3,3\rangle_{LR}t}}{12+4x}[$$

$$mod\left(\tilde{T}_{s_2}(t)\right) = \frac{1-e^{-4t}}{5+4e^{-4t}}.$$

1.7. Average number of customers in the queue over a period of time t.

$$\tilde{N}_{q_2}(t) = \tilde{T}_{s_2}(t).\lambda = \frac{\tilde{\lambda}_2}{\tilde{\mu}-\tilde{\lambda}}\left(\frac{\tilde{\mu}-\tilde{\lambda}_1+\tilde{\lambda}_2\tilde{\lambda}_1}{\tilde{\mu}-\tilde{\lambda}_1}\right)\frac{(1-x)}{\tilde{\lambda}+(\tilde{\mu}-\tilde{\lambda})x}. \ \tilde{\lambda}$$

$$= \langle\frac{1-x}{5+4x}, \frac{362+165x-268x^2}{(5+4x)(392+456x)}, \frac{288+133x-88x^2}{(5+4x)(12+4x)}\rangle_{LR}.\langle 5,2,2\rangle_{LR}$$

$$= \langle\frac{1-x}{5+4x}.5, \frac{1-x}{5+4x}.2 + 5.\frac{362+165x-268x^2}{(5+4x)(392+456x)} - 2.\frac{362+165x-268x^2}{(5+4x)(392+456x)}, \frac{1-x}{5+4x}.2 + 5.\frac{288+133x-88x^2}{(5+4x)(12+4x)} + 2.\frac{288+133x-88x^2}{(5+4x)(12+4x)}\rangle_{LR}$$

$$= \langle\frac{5-5x}{5+4x}, \frac{1870+623x-1716x^2}{(5+4x)(392+456x)}, \frac{2040+915x-624x^2}{(5+4x)(12+4x)}\rangle_{LR},$$

$$supp\left(\tilde{N}_{q_2}(t)\right) =]\frac{90-303x-564x^2}{(5+4x)(392+456x)}, \frac{2100+875x-644x^2}{(5+4x)(12+4x)}[,$$

$$mod\left(\tilde{N}_{q_2}(t)\right) = \frac{5-5e^{-4t}}{5+4e^{-4t}}.$$

1.8. Average waiting time for customers in the queue for a given duration t.

$$\tilde{T}_{q_2}(t) = \tilde{T}_{s_2}(t) - \frac{1}{\tilde{\mu}} = \frac{\tilde{\lambda}_2}{\tilde{\mu}-\tilde{\lambda}}\left(\frac{\tilde{\mu}-\tilde{\lambda}_1+\tilde{\lambda}_2\tilde{\lambda}_1}{\tilde{\mu}-\tilde{\lambda}_1}\right)\frac{(1-x)}{\tilde{\lambda}+(\tilde{\mu}-\tilde{\lambda})x} - \frac{1}{\tilde{\mu}},$$

$$\tilde{T}_{q_2}(t) = \frac{\langle 24-24x, 18+x, 36+x\rangle_{LR}}{\langle 102+96x, 108+92x, 272+360x\rangle_{LR}} - \frac{1}{\langle 9,1,1\rangle_{LR}}$$

$$= \frac{\langle 24-24x, 18+x, 36+x\rangle_{LR}\times\langle 9,1,1\rangle_{LR} - \langle 102+96x, 108+92x, 272+360x\rangle_{LR}}{\langle 102+96x, 108+92x, 272+360x\rangle_{LR}\times\langle 9,1,1\rangle_{LR}}$$

$$= \langle\frac{19-52x}{153+144x}, \frac{725628+236832x-855936x^2}{(918+864x)(3740+4560x)}, \frac{1338444+2095092x+786240x^2}{(918+864x)(-48+32x)}\rangle_{LR},$$

$$supp\left(\tilde{T}_{q_2}(t)\right) =]\frac{-326-656x}{(3740+4560x)}, \frac{605-234x}{(-48+32x)}[,$$

$$mod\left(\tilde{T}_{q_2}(t)\right) = \frac{19-52e^{-4t}}{153+144e^{-4t}}.$$

2. Graphical representations

If t varies from 0 to ∞, the open modes and intervals describing the supports of the fuzzy performance parameters in the priority transient regime allow the plots in Figure 1 - 4.

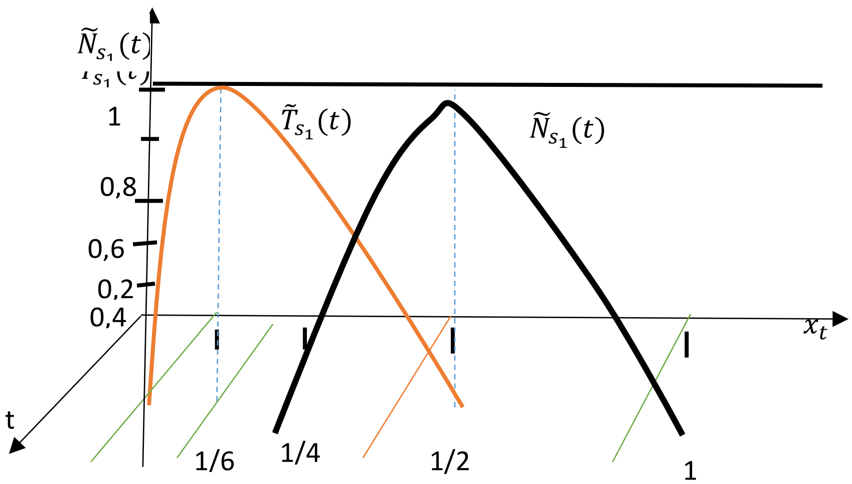

Figure 1. Fuzzy membership function graphs of average number $\widetilde{N}_{s_1}(t)$ and average waiting time $\widetilde{T}_{s_1}(t)$ of priority customers in the system.

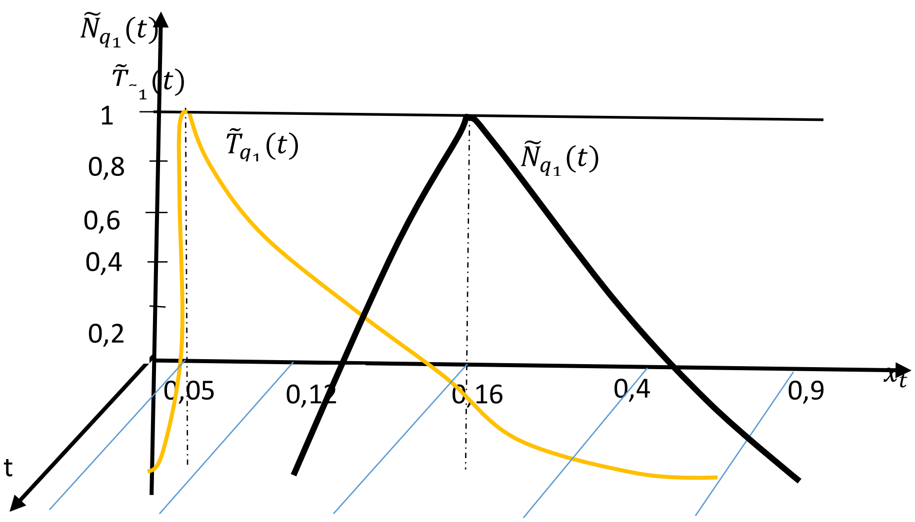

Figure 2. Membership function graphs of the fuzzy functions average number $\widetilde{N}_{q_1}(t)$ and average waiting time $\widetilde{T}_{q_1}(t)$ of priority customers in the queue.

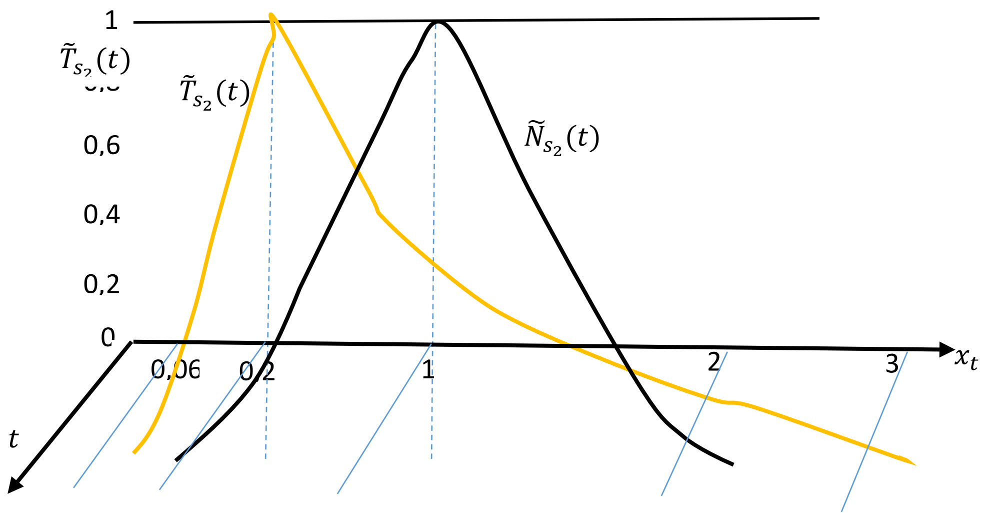

Figure 3. Membership function graphs of the fuzzy functions average number $\tilde{N}_{s_2}(t)$ and average waiting times $\tilde{T}_{s_2}(t)$ of non-priority customers in the system.

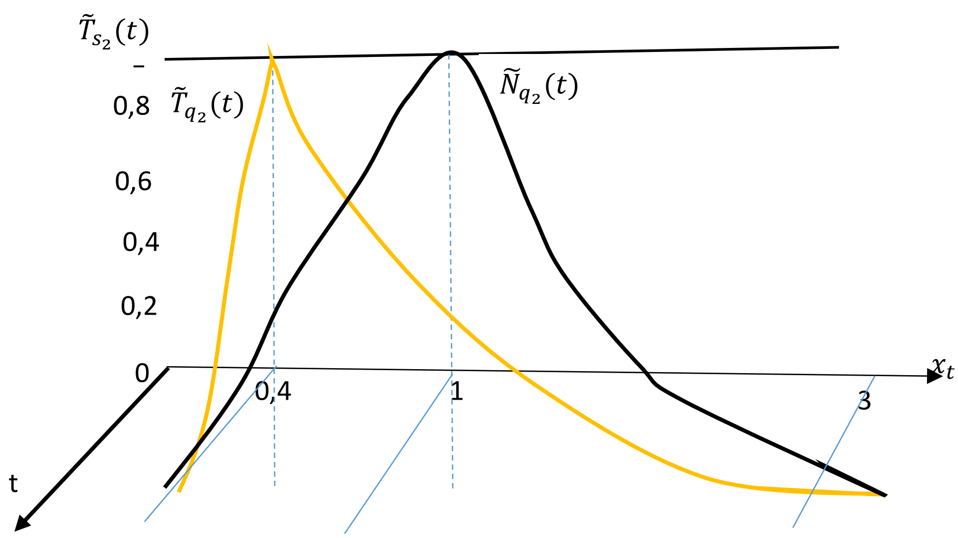

Figure 4. Membership function graphs of the fuzzy functions average number $\tilde{N}_{q_2}(t)$ and average waiting times $\tilde{T}_{q_2}(t)$ of non-priority customers in the queue.

3. DISCUSSION

The conditions of stability of the system having been established: $\lambda_1 < \mu$; $\lambda_2 < \mu$ *and* $\lambda < \mu$, they allow us to evaluate the performance parameters of each category of customers and push us to issue a point of view on the results obtained. Regarding priority customers, we observe the following:

The modal value is $mod\left(\tilde{N}_{s1}(t)\right) = \frac{1}{2} - \frac{e^{-6t}}{2}$, and The support is $supp\left(\tilde{N}_{s1}(t)\right) = \left]\frac{1-2e^{-\langle 6,2,2\rangle_{LR}t}}{4}, \frac{2-e^{-\langle 6,2,2\rangle_{LR}t}}{2}\right[$. In the limit when t→∞, the average number of priority customers in the system varies approximately between 0.25 and 1 customer with a most likely value of 0.5 customers.

As for their waiting time in the system, the modal value is $mod\left(\tilde{T}_{s1}(t)\right) = \frac{1-e^{-6t}}{6+12e^{-6t}}$ and support $\left(\tilde{T}_{s1}(t)\right) = \left]\frac{1-2e^{-\langle 6,2,2\rangle_{LR}t}}{16+32x}, \frac{2-e^{-\langle 6,2,2\rangle_{LR}t}}{4+8x}\right[$. At limit when t→∞, this time varies between 4 and 30 minutes and its most possible value is 10 minutes.

Analyzing the situation of non-priority customers, the following is revealed:

The modal values of the average number of non-priority customers in the system and support are as follows: $mod\left(\tilde{N}_{s_2}(t)\right) = 1 - e^{-4t}$ and $supp\left(\tilde{N}_{s_2}(t)\right) \left]\frac{1-29e^{-\langle 4,3,3\rangle_{LR}t}}{15}, \frac{23+24e^{-\langle 4,3,3\rangle_{LR}t}}{8}\right[$. If $t \to \infty$, this average number is approximately between 0.6 and 3 customers, with a maximum value of 1 customer.

The waiting time modal value is $\left(\tilde{T}_{s_2}(t)\right) = \frac{1-e^{-4t}}{5+4e^{-4t}}$, and the support is $pp\left(\tilde{T}_{s_2}(t)\right) = \left]\frac{10-47e^{-\langle 4,3,3\rangle_{LR}t}}{392+456x}, \frac{60-23e^{-\langle 4,3,3\rangle_{LR}t}}{12+4x}\right[$. When $t \to \infty$, this time is approximately between 20 and 300 minutes, with a likely value of 12 minutes.

From the above, the number and average time in the system of non-priority customers are higher than that of priority customers, simply because the services of non-priority customers are highly disrupted by the arrivals and service times of priority customers, given that priority is non-preemptive.

4. CONCLUSION

At the end of this reflection on the performance measures of a queue with absolute priority in a transient regime in a fuzzy environment, the FM/FM/1 model, our objective was to calculate its parameters using one of the fuzzy arithmetic methods. We used the L-R method applied to triangular fuzzy numbers to achieve this. This enabled us to calculate the membership functions of the various performance parameters of priority and non-priority customers, and to deduce the supports and modal values. The average number of customers and the average waiting time expected in the system during a period t are derived, and the results are given in L-R. Supported by a numerical example with graphical representations of the performance measures in a three-dimensional space, our results, compared with other previous publications in the same field, add value by the fact that we treat, in addition to the performance parameters of priority customers, the performance measures of non-priority customers under transient conditions in a fuzzy environment. This marks the originality of this paper. Sooner or later, this approach may be extended to other queuing models, such as FM/FM/C and FM/G/1.

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