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On Randers Change of a Generalized Exponential Metric

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Abstract

In this paper, we study the properties of a special (α,β)-metric $e^{k_1\frac{\beta}{\alpha}}+\beta e^{k_2\frac{\beta}{\alpha}}$, the Randers change of the generalized exponential metric. We find the necessary and sufficient condition for this metric to be locally projectively flat and we also prove the conditions for this metric to be of the Berwald and Douglas type.

Keywords: Berwald space; Douglas space; Finsler space; (α,β)-metric; projectively flat.

Abstrak

Pada artikel ini akan dipelajari sifat-sifat khusus dari (α,β)-metric $e^{k_1\frac{\beta}{\alpha}}+\beta e^{k_2\frac{\beta}{\alpha}}$, *perubahan* Randers *dari metrik eksponensial umum. Kami menemukan syarat perlu dan cukup agar metrik ini menjadi datar secara lokal dan kami juga membuktikan syarat agar metrik ini bertipe* Berwald *dan* Douglas.

Kata Kunci: *ruang* Berwald; *ruang* Douglas; *ruang* Finsler; (α,β)-metric; projectively flat.

2020MSC: 53B20.

1. INTRODUCTION

Graph In 1972, M. Matsumoto [1] [2] introduced the notion of the (α,β)-metric on the basis of the Randers metric. There are several types of (α,β)-metrics, such as the Randers metric $\alpha+\beta$ [3], Kropina metric $\frac{\alpha^2}{\beta}$ [1] [4], generalized Kropina metric $\frac{\alpha^{m+1}}{\beta^m}$ $(m\neq 0,-1)$ [5] and Matsumoto metric $\frac{\alpha^2}{\alpha-\beta}$ [4]. In these metrics, $\alpha(x,y)=(a_{ij}(x)y^iy^j)^{1/2}$ is a Riemannian metric and $\beta(x,y)=b_i(x)y^i$ is a differential one form [6].

In 2013, G. Shanker and Ravindra [7] introduced a special (α,β)-metric $\alpha e^{\beta/\alpha}+\beta$ and considered it a Randers change in the exponential metric. Now, we generalize this concept by introducing an (α,β)-metric

$$\mathrm{F}(\alpha,\beta)=\alpha e^{k_1\frac{\beta}{\alpha}}+\beta e^{k_2\frac{\beta}{\alpha}}, \tag{1}$$

where k_1 and k_2 are some constants and are referred to as the Randers changes of the generalized exponential metric.

In this paper, we have studied the properties of this (α,β)- metric, identified the necessary and sufficient conditions for the metric to be locally projectively flat and determined the conditions for this metric to be of the Berwald and Douglas type. A deep study of the projectively flat Finsler metric was also carried out by authors [5] [8] [9] [10] [11] [12] and [13].

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2. PRELIMINARIES

Let $\alpha = (a_{ij}(x)y^i y^j)^{1/2}$ be a Riemannian metric, $\beta = b_i y^j$ be a 1-form and $F = \alpha\phi(s)$, where $s = \frac{\beta}{\alpha}$, $\phi = \phi(s)$ be a positive C^∞ function defined in a neighborhood of the origin $s = 0$. Since we know that $F = \alpha\phi\left(\frac{\beta}{\alpha}\right)$ is a Finsler metric for any α and β with $b = \parallel \beta \parallel_\alpha < b_0$ iff

$$\phi(s) > 0, \phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0, (|s| \leq b < b_0).$$

Replacing b by s, we obtain

$$\phi(s) - s\phi'(s) > 0, (|s| < b_0).$$

Let G^i and G^i_α denote the spray coefficients of F and α respectively, defined as

$$G^i = \frac{g^{il}}{4}\left\{[F^2]_{x^k y^l}y^k - [F^2]_{x^k}\right\}, G^i_\alpha = \frac{a^{il}}{4}\left\{[\alpha^2]_{x^k y^l}y^k - [\alpha^2]_{x^k}\right\},$$

where $g_{ij} = \frac{1}{2}[F^2]_{y^i y^j}$ and

$$\left(a^{ij}\right) = \left(a_{ij}\right)^{-1}, F_{x^k} = \frac{\partial F}{\partial x^k}, F_{y^k} = \frac{\partial F}{\partial y^k}.$$

For an (α, β)-metric $L(\alpha, \beta)$, the space $\mathbb{R}^n = (M^n, \alpha)$ is known as the Riemannian space associated with the Finsler space $F^n = (M^n, L(\alpha, \beta))$. The covariant differentiation with respect to the Levi-Civita connection $\gamma^i_{jk}(x)$ of $\mathbb{R}^n$ is denoted by $(;)$. Now, we have the following [14]:

Lemma 2.1. The spray coefficients G^i are related to G^i_α by

$$G^i = G^i_\alpha + \alpha Q s^i_0 + J(-2\alpha Q s_0 + r_{00})\frac{y^i}{\alpha} + H(-2\alpha Q s_0 + r_{00})\left\{b^i - \frac{y^i}{\alpha}\right\}, \tag{2}$$

where

$$Q = \frac{\phi'}{\phi - s\phi'},$$
$$J = \frac{(\phi - s\phi')\phi'}{2\phi\left((\phi - s\phi') + (b^2 - s^2)\phi''\right)},$$
$$H = \frac{\phi''}{2\left((\phi - s\phi') + (b^2 - s^2)\phi''\right)}$$

where $s_{l0} = s_{li}y^i, s_0 = s_{l0}b^l, r_{00} = r_{ij}y^i y^j, r_{ij} = \frac{1}{2}\left(b_{i;j} + b_{j;i}\right), s_{ij} = \frac{1}{2}\left(b_{i;j} - b_{j;i}\right), r^i_j = a^{ir}r_{rj}, s^i_j = a^{ir}s_{rj}, r_j = b_r r^r_j, s_j = b_r s^r_j, b^i = a^{ir}b_r$ and $b^2 = a^{rs}b_r b_s$. It is well-known that [15] a Finsler metric $F = F(x, y)$ on an open subset $U \subset \mathbb{R}^n$ is projectively flat if and only if

$$F_{x^k y^l}y^k - F_{x^l} = 0. \tag{3}$$

By equation (3), we have the following lemma [9].

Lemma 2.2. An (α,β)-metric $F=\alpha\phi(s)$, where $s=\frac{\beta}{\alpha}$, is projectively flat on an open subset $U\subset\mathbb{R}^n$ if and only if

$$(a_{ml}\alpha^2-y_m y_l)G_\alpha^m+\alpha^3 Q s_{l0}+H\alpha(-2\alpha Q s_0+r_{00})(b_l\alpha-s y_l)=0. \tag{4}$$

The functions $G^i(x,y)$ of F^n with an (α,β)-metric can be written in the form [16]

$$2G^i=\gamma_{00}^i+2B^i, \tag{5}$$

$$B^i=\frac{\alpha L_\beta}{L_\alpha}s_0^i+C^*\left\{\frac{\beta L_\beta}{\alpha L}y^i-\frac{\alpha L_{\alpha\alpha}}{L_\alpha}\left(\frac{1}{\alpha}y^i-\frac{\alpha}{\beta}b^i\right)\right\}. \tag{6}$$

provided $\beta^2+L_\alpha+\alpha\gamma^2 L_{\alpha\alpha}\neq 0$, where $\gamma^2=b^2\alpha^2-\beta^2, L_\alpha=\frac{\partial L}{\partial\alpha}, L_\beta=\frac{\partial L}{\partial\beta}, L_{\alpha\alpha}=\frac{\partial L_\alpha}{\partial\alpha}$, and $s_0^i=s_j^i y^j$. Now we put

$$C^*=\frac{\alpha\beta(r_{00}L_\alpha-2\alpha s_0 L_\beta)}{2(\beta^2 L_\alpha+\alpha\gamma^2 L_{\alpha\alpha})}.$$

We denote the homogeneous polynomials in (y^i) of degree r by $hp(r)$ for brevity. For example, γ_{00}^i is $hp(2)$. From equation (5), the Berwald connection $B\Gamma=(G_{jk}^i,G_j^i,0)$ of F^n with an (α,β)-metric is given by [16]

$$G_j^i=\dot{\partial}_j G^i=\gamma_{0j}^i+B_j^i$$
$$G_{jk}^i=\dot{\partial}_k G_j^i=\gamma_{jk}^i+B_{jk}^i,$$

where $B_j^i=\dot{\partial}_j B^i$ and $B_{jk}^i=\dot{\partial}_k B_j^i$. According to [16], B_{jk}^i are determined by

$$L_\alpha B_{ji}^k y^j y_t+\alpha L_\beta(B_{ji}^k b_t-b_{j;i})y^j=0, \tag{7}$$

where $y_k=a_{ik}y^i$. A Finsler space F^n with an (α,β)-metric is a Douglas space if and only if $B^{ij}\equiv B^i y^j-b^j y^i$ is $hp(3)$ [17]. From equation (6), B^{ij} can be written as follows:

$$B^{ij}=\frac{\alpha L_\beta}{L_\alpha}(s_0^i y^j-s_0^j y^i)+\frac{\alpha^2 L_{\alpha\alpha}}{\beta L_\alpha}C^*(b^i y^j-b^j y^i). \tag{8}$$

3. RESULTS AND DISCUSSION

3.1. Projectively flat (α,β)-metric

In this context, we consider the metric $F=\alpha e^{k_1\frac{\beta}{\alpha}}+\beta e^{k_2\frac{\beta}{\alpha}}$ which is obtained by the Randers change of the generalized exponential metric

$$F=\alpha\phi(s),\phi(s)=(e^{k_1 s}+s e^{k_2 s}), s=\frac{\beta}{\alpha}. \tag{9}$$

Let $b_0>0$ be the largest number satisfying

$$\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0, (|s| \le b < b_0),$$

that is,

$$e^{k_1 s}(1 - sk_1 + b^2k_1^2 - s^2k_1^2) + e^{k_2 s}(b^2sk_2^2 - s^2k_2 + 2bk_2 - s^3k_2^2 - 2k_2s^2) > 0, (|s| \le b < b_0).$$

Lemma 3.1. $F = \alpha e^{k_1\frac{\beta}{\alpha}} + \beta e^{k_2\frac{\beta}{\alpha}}$ is a Finsler metric, iff $\| \beta \|_\alpha < 1$.

Proof.
If $F = \alpha e^{k_1\frac{\beta}{\alpha}} + \beta e^{k_2\frac{\beta}{\alpha}}$ is a Finsler metric, then

$$e^{k_1 s}(1 - sk_1 + b^2k_1^2 - s^2k_1^2) + e^{k_2 s}(b^2sk_2^2 - s^2k_2 + 2bk_2 - s^3k_2^2 - 2k_2s^2) > 0, (|s| \le b < b_0).$$

Let $s = b$, then we obtain $b < 1, \forall b < b_0$. Let $b \to b_0$, then $b_0 < 1$. Therefore $\| \beta \|_\alpha < 1$. Now, if $|s| \le b < 1$ then

$$e^{k_1 s}(1 - sk_1 + b^2k_1^2 - s^2k_1^2) + e^{k_2 s}(b^2sk_2^2 - s^2k_2 + 2bk_2 - s^3k_2^2 - 2k_2s^2) \ge e^{k_1 s}(1 - sk_1) + e^{k_2 s}(2bk_2 - 3k_2s^2) > 0 \quad (\text{since } b^2 - s^2 \ge 0).$$

Thus $F = \alpha e^{k_1\frac{\beta}{\alpha}} + \beta e^{k_2\frac{\beta}{\alpha}}$ is a Finsler metric. By Lemma (2.1), the spray coefficients G^i of F are written by equation (2) as

$$Q = \frac{k_1e^{k_1 s} + k_2se^{k_2 s} + e^{k_2 s}}{e^{k_1 s} - k_1se^{k_1 s} - k_2s^2e^{k_2 s}}, \qquad s = \frac{\beta}{\alpha},$$

$$J = \frac{(1 - sk_1)k_1e^{2k_1 s} + \{(1 - sk_1)sk_2 + (1 - sk_1) - s^2k_1k_2\}e^{(k_1+k_2)s} - s^2k_2(1 + sk_2)e^{2k_2 s}}{2(e^{k_1 s} + se^{k_2 s})\{[(1 - sk_1) + (b^2 - s^2)k_1]e^{k_1 s} + [(b^2 - s^2)k_2^2s + 2b^2k_2 - 3s^2k_2]e^{k_2 s}\}},$$

$$H = \frac{k_1^2e^{k_1 s} + k_2^2se^{k_2 s} + 2k_2e^{k_2 s}}{2\{[(1 - sk_1) + (b^2 - s^2)k_1]e^{k_1 s} + [(b^2 - s^2)k_2^2s + 2b^2k_2 - 3s^2k_2]e^{k_2 s}\}}.$$

Then equation (4) is reduced to the following form:

$$(a_{ml}\alpha^2 - y_my_l)G^m_\alpha + \alpha^3\frac{k_1e^{k_1 s} + k_2se^{k_2 s} + e^{k_2 s}}{e^{k_1 s} - k_1se^{k_1 s} - k_2s^2e^{k_2 s}}s_{l0} +$$

$$\frac{\alpha(k_1^2e^{k_1 s} + k_2^2se^{k_2 s} + 2k_2e^{k_2 s})}{2\{[(1-sk_1) + (b^2-s^2)k_1]e^{k_1 s} + [(b^2-s^2)k_2^2s + 2b^2k_2 - 3s^2k_2]e^{k_2 s}\})}\left[-2\alpha\frac{(k_1e^{k_1 s} + k_2se^{k_2 s} + e^{k_2 s})}{(e^{k_1 s} - k_1se^{k_1 s} - k_2s^2e^{k_2 s})}s_0 + r_{00}\right]\times$$

$$(b_l\alpha - sy_l) = 0, s = \frac{\beta}{\alpha}. \tag{10}$$

Now we use the following result [18]:

Lemma 3.2. If $(a_{ml}\alpha^2 - y_my_l)G^m_\alpha = 0$, then α is projectively flat.

Proof.
If $(a_{ml}\alpha^2 - y_my_l)G^m_\alpha = 0$, then we have

$$\alpha^2 a_{ml} G_\alpha^m = y_m G_\alpha^m y_l.$$

Hence, if there is an $\eta = \eta(x, y)$ such that $y_m G_\alpha^m = \alpha^2 \eta$, we find

$$a_{ml} G_\alpha^m = \eta y_l. \tag{11}$$

Contracting equation (11) with a^{il} yields $G_\alpha^i = \eta y^i$, and hence α is projectively flat.

Theorem 3.3. The (α, β)-metric $F = \alpha e^{k_1 \frac{\beta}{\alpha}} + \beta e^{k_2 \frac{\beta}{\alpha}}$ is locally projectively flat iff β is parallel with respect to α and α is locally projectively flat, i.e., of constant curvature.

Proof.

Suppose, that F is locally projectively flat. First, we rewrite equation (10) as a polynomial in y^i and α, then we obtain

$$\{-2\alpha^2\beta[(1+k_1)+b^2] + 2\beta^3 k_1\} e^{k_1\frac{\beta}{\alpha}} (a_{ml}\alpha^2 - y_m y_l) G_\alpha^m + (2\alpha^6 + 2b^2\alpha^6 - 2\alpha^4\beta^2)\left[k_1 e^{k_1\frac{\beta}{\alpha}} + \left(1 + k_2\frac{\beta}{\alpha}\right) e^{k_2\frac{\beta}{\alpha}}\right] s_{l0} - (2\alpha^4 s_0 \left[k_1 e^{k_1\frac{\beta}{\alpha}} + \left(1 + k_2\frac{\beta}{\alpha}\right) e^{k_2\frac{\beta}{\alpha}}\right] - \alpha^2\beta e^{k_1\frac{\beta}{\alpha}} r_{00})(b_l\alpha^2 - \beta y_l)\} + \alpha\{(2\alpha^2 + 2b^2\alpha^2) e^{k_1\frac{\beta}{\alpha}} (a_{ml}\alpha^2 - y_m y_l) G_\alpha^m - 2\alpha^4\beta\left[k_1 e^{k_1\frac{\beta}{\alpha}} + \left(1 + k_2\frac{\beta}{\alpha}\right) e^{k_2\frac{\beta}{\alpha}}\right] s_{l0} + \alpha^2 e^{k_1\frac{\beta}{\alpha}} r_{00}(b_l\alpha^2 - \beta y_l) - k_2\beta^2 r_{00} e^{k_2\frac{\beta}{\alpha}} (b_l\alpha^2 - \beta y_l)\} = 0,$$

$$\text{or } U + \alpha V = 0, \tag{12}$$

where

$$U = \{-2\alpha^2\beta[(1+k_1)+b^2] + 2\beta^3 k_1\} e^{k_1\frac{\beta}{\alpha}} (a_{ml}\alpha^2 - y_m y_l) G_\alpha^m + (2\alpha^6 + 2b^2\alpha^6 - 2\alpha^4\beta^2) \times \left[k_1 e^{k_1\frac{\beta}{\alpha}} + \left(1 + k_2\frac{\beta}{\alpha}\right) e^{k_2\frac{\beta}{\alpha}}\right] s_{l0} - (2\alpha^4 s_0 \left[k_1 e^{k_1\frac{\beta}{\alpha}} + \left(1 + k_2\frac{\beta}{\alpha}\right) e^{k_2\frac{\beta}{\alpha}}\right] + \alpha^2\beta e^{k_1\frac{\beta}{\alpha}} r_{00})(b_l\alpha^2 - \beta y_l)\},$$

and

$$V = 2\alpha^2(1+b^2) e^{k_1\frac{\beta}{\alpha}} (a_{ml}\alpha^2 - y_m y_l) G_\alpha^m - 2\alpha^4\beta\left[k_1 e^{k_1\frac{\beta}{\alpha}} + \left(1 + k_2\frac{\beta}{\alpha}\right) e^{k_2\frac{\beta}{\alpha}}\right] s_{l0} + (\alpha^2 e^{k_1\frac{\beta}{\alpha}} - \beta^2 k_2 e^{k_2\frac{\beta}{\alpha}})\, r_{00}(b_l\alpha^2 - \beta y_l).$$

Now, equation (12) is a polynomial in y^i, such that U and V are rational in y^i and α is irrational. Therefore, we must have $U = 0$ and $V = 0$ which implies that

$$\{-2\alpha^2\beta[(1+k_1)+b^2] + 2\beta^3 k_1\} e^{k_1\frac{\beta}{\alpha}} (a_{ml}\alpha^2 - y_m y_l) G_\alpha^m = \left(2\alpha^4 s_0 \left[k_1 e^{k_1\frac{\beta}{\alpha}} + \left(1 + k_2\frac{\beta}{\alpha}\right) e^{k_2\frac{\beta}{\alpha}}\right] + \alpha^2\beta e^{k_1\frac{\beta}{\alpha}} r_{00}\right) \times (b_l\alpha^2 - \beta y_l) - (2\alpha^6 + 2b^2\alpha^6 - 2\alpha^4\beta^2)\left[k_1 e^{k_1\frac{\beta}{\alpha}} + \left(1 + k_2\frac{\beta}{\alpha}\right) e^{k_2\frac{\beta}{\alpha}}\right] s_{l0}, \tag{13}$$

and

$$2\alpha^2(1+b^2) e^{k_1\frac{\beta}{\alpha}} (a_{ml}\alpha^2 - y_m y_l) G_\alpha^m - 2\alpha^4\beta\left[k_1 e^{k_1\frac{\beta}{\alpha}} + \left(1 + k_2\frac{\beta}{\alpha}\right) e^{k_2\frac{\beta}{\alpha}}\right] s_{l0} - (\alpha^2 e^{k_1\frac{\beta}{\alpha}} - \beta^2 k_2 e^{k_2\frac{\beta}{\alpha}})\, r_{00}(b_l\alpha^2 - \beta y_l), \tag{14}$$

respectively. Contracting equations (13) and (3.6) with b^l, we have

$$\{-2\alpha^2\beta[1+k_1+b^2]+2\beta^3k_1\}e^{k_1\frac{\beta}{\alpha}}(b_m\alpha^2-y_m\beta)G_\alpha^m$$
$$=-2\alpha^6 s_0\left[k_1e^{k_1\frac{\beta}{\alpha}}+\left(1+k_2\frac{\beta}{\alpha}\right)e^{k_2\frac{\beta}{\alpha}}\right]+\alpha^2\beta e^{k_1\frac{\beta}{\alpha}}r_{00}(b^2\alpha^2-\beta^2)$$

$$2\alpha^2(1+b^2)e^{k_1\frac{\beta}{\alpha}}(b_m\alpha^2-y_m\beta)G_\alpha^m$$
$$=2\alpha^4\,\beta\left[k_1e^{k_1\frac{\beta}{\alpha}}+\left(1+k_2\frac{\beta}{\alpha}\right)e^{k_2\frac{\beta}{\alpha}}\right]s_0-(\alpha^2e^{k_1\frac{\beta}{\alpha}}-\beta^2k_2e^{k_2\frac{\beta}{\alpha}})r_{00}(b^2\alpha^2-\beta^2) \tag{15}$$

$$2(1+b^2)e^{k_1\frac{\beta}{\alpha}}(b_m\alpha^2-y_m\beta)G_\alpha^m$$
$$=2\alpha^2\,\beta\left[k_1e^{k_1\frac{\beta}{\alpha}}+\left(1+k_2\frac{\beta}{\alpha}\right)e^{k_2\frac{\beta}{\alpha}}\right]s_0-\left(e^{k_1\frac{\beta}{\alpha}}-\frac{\beta^2}{\alpha^2}k_2e^{k_2\frac{\beta}{\alpha}}\right)r_{00}(b^2\alpha^2-\beta^2) \tag{16}$$

Multiplying equation (16) by $\alpha^2\beta$ and adding the result obtained to equation (15), we get

$$\beta\left[2k_1(-\alpha^2+\beta^2)(b_m\alpha^2-y_m\beta)e^{k_1\frac{\beta}{\alpha}}G_\alpha^m-k_2\beta^2r_{00}e^{k_2\frac{\beta}{\alpha}}(b^2\alpha^2-\beta^2)\right]$$
$$=2\alpha^4(-\alpha^2+\beta^2)\left[k_1e^{k_1\frac{\beta}{\alpha}}+\left(1+k_2\frac{\beta}{\alpha}\right)e^{k_2\frac{\beta}{\alpha}}\right]s_0 \tag{17}$$

The polynomial $2\alpha^4(-\alpha^2+\beta^2)\left[k_1e^{k_1\frac{\beta}{\alpha}}+\left(1+k_2\frac{\beta}{\alpha}\right)e^{k_2\frac{\beta}{\alpha}}\right]$ is not divisible by β and β is not divisible by

$$2\alpha^4(-\alpha^2+\beta^2)\left[k_1e^{k_1\frac{\beta}{\alpha}}+\left(1+k_2\frac{\beta}{\alpha}\right)e^{k_2\frac{\beta}{\alpha}}\right].$$

Thus s_0 is divisible by β and $[2k_1(-\alpha^2+\beta^2)(b_m\alpha^2-y_m\beta)e^{k_1\frac{\beta}{\alpha}}G_\alpha^m-k_2\beta^2r_{00}e^{k_2\frac{\beta}{\alpha}}(b^2\alpha^2-\beta^2)]$ is divisible by $2\alpha^4(-\alpha^2+\beta^2)\left[k_1e^{k_1\frac{\beta}{\alpha}}+\left(1+k_2\frac{\beta}{\alpha}\right)e^{k_2\frac{\beta}{\alpha}}\right]$. Hence, there exist scalar functions $\tau=\tau(x),\chi=\chi(x)$ such that

$$s_0=\tau\beta, \tag{18}$$

$$2k_1(-\alpha^2+\beta^2)(b_m\alpha^2-y_m\beta)e^{k_1\frac{\beta}{\alpha}}G_\alpha^m-k_2\beta^2r_{00}e^{k_2\frac{\beta}{\alpha}}(b^2\alpha^2-\beta^2)$$
$$=\chi\left\{2\alpha^4(-\alpha^2+\beta^2)\left[k_1e^{k_1\frac{\beta}{\alpha}}+\left(1+k_2\frac{\beta}{\alpha}\right)e^{k_2\frac{\beta}{\alpha}}\right]\right\}. \tag{19}$$

Then, equation (17) reduces to

$$\beta\chi\left\{2\alpha^4(-\alpha^2+\beta^2)\left[k_1e^{k_1\frac{\beta}{\alpha}}+\left(1+k_2\frac{\beta}{\alpha}\right)e^{k_2\frac{\beta}{\alpha}}\right]\right\}=2\alpha^4(-\alpha^2+\beta^2)\left[k_1e^{k_1\frac{\beta}{\alpha}}+\left(1+k_2\frac{\beta}{\alpha}\right)e^{k_2\frac{\beta}{\alpha}}\right]\tau\beta.$$

Thus $\tau=\chi$. Hence, equation (15) becomes

$$\chi\left\{2\alpha^4(-\alpha^2+\beta^2)\left[k_1e^{k_1\frac{\beta}{\alpha}}+\left(1+k_2\frac{\beta}{\alpha}\right)e^{k_2\frac{\beta}{\alpha}}\right]\right\}\{-2\alpha^2\beta[1+k_1+b^2]+2\beta^3k_1+2k_1\beta\alpha^2\}=$$
$$\left\{(2\alpha^2\beta[1+k_1+b^2]-2\beta^3k_1)k_2\beta^2e^{k_2\frac{\beta}{\alpha}}+\alpha^2\beta e^{k_1\frac{\beta}{\alpha}}(-\alpha^2+\beta^2)2k_1\right\}r_{00}(b^2\alpha^2-\beta^2). \tag{20}$$

Since $(b^2\alpha^2-\beta^2)$ is not divisible by

$$\frac{\left\{2\alpha^4(-\alpha^2+\beta^2)\left[k_1e^{k_1\frac{\beta}{\alpha}}+\left(1+k_2\frac{\beta}{\alpha}\right)e^{k_2\frac{\beta}{\alpha}}\right]\right\}\{-2\alpha^2\beta[1+k_1+b^2]+2\beta^3k_1+2k_1\beta\alpha^2\}}{\left\{(2\alpha^2\beta[1+k_1+b^2]-2\beta^3k_1)k_2\beta^2e^{k_2\frac{\beta}{\alpha}}+\alpha^2\beta e^{k_1\frac{\beta}{\alpha}}(-\alpha^2+\beta^2)2k_1\right\}},$$

it follows from (20) that $\chi=0$. By (18), (19) and (20), we get

$$s_0=0, \tag{21}$$
$$r_{00}=0, \tag{22}$$

and $2k_1(-\alpha^2+\beta^2)(b_m\alpha^2-y_m\beta)e^{k_1\frac{\beta}{\alpha}}G_\alpha^m-k_2\beta^2r_{00}e^{k_2\frac{\beta}{\alpha}}(b^2\alpha^2-\beta^2)=0.$

Using equation (22) in the above equation, we obtain

$$(b_m\alpha^2-y_m\beta)e^{k_1\frac{\beta}{\alpha}}G_\alpha^m=0. \tag{23}$$

Substituting equations(22) and (23) into (14), we find

$$s_{l0}=0. \tag{24}$$

Using equation (23) and lemma (3.2), givesα is projectively flat. Using equations (22) and (24),we get $b_{i;j}=0$. i.e, β is parallel with respect to α. Conversely, if β is parallel with respect to α and α is locally projectively flat, then by Lemma (2.2), we can easily see that F is locally projectively flat.

3.2. Berwald and Douglas Spaces

In this part, we find the condition that a Finsler space F^n withthe (α,β)-metric (9) is a Berwald space. In n-dimensional Finsler space F^n with the (α,β)-metric (9), we have

$$L_\alpha=\frac{(\alpha-k_1\beta)e^{k_1\frac{\beta}{\alpha}}}{\alpha},\ L_\beta=k_1e^{k_1\frac{\beta}{\alpha}}+e^{k_2\frac{\beta}{\alpha}}\left(1+k_2\frac{\beta}{\alpha}\right),$$
$$L_{\alpha\alpha}=\frac{1}{\alpha}\left(\frac{k_1\beta}{\alpha}\right)^2e^{k_1\frac{\beta}{\alpha}},L_{\beta\beta}=\frac{{k_1}^2}{\alpha}e^{k_1\frac{\beta}{\alpha}}+\frac{k_2}{\alpha}\left(\frac{2\alpha+k_2\beta}{\alpha}\right)e^{k_2\frac{\beta}{\alpha}} \tag{25}$$

Substituting (25) in (7), we find

$$\alpha e^{k_1\frac{\beta}{\alpha}}B_{ji}^ty^jy_t-k_1\beta e^{k_1\frac{\beta}{\alpha}}B_{ji}^ty^jy_t+\left\{\alpha^2\left(k_1e^{k_1\frac{\beta}{\alpha}}+e^{k_2\frac{\beta}{\alpha}}\right)+\beta\alpha k_2e^{k_2\frac{\beta}{\alpha}}\right\}\left(B_{ji}^tb_t-b_{j;i}\right)y^j=0. \tag{26}$$

Let F^n be a Berwald space, i.e., $G_{jk}^i=G_{jk}^i(x)$. Then we obtain $B_{ji}^t=B_{ji}^t(x)$. Since α is irrational in (y^i), hence from equation (26), we can write

$$e^{k_1\frac{\beta}{\alpha}}B_{ji}^ty^jy_t=0, \tag{27}$$

$$-k_1\beta e^{k_1\frac{\beta}{\alpha}}B_{ji}^ty^jy_t+\left\{\alpha^2\left(k_1e^{k_1\frac{\beta}{\alpha}}+e^{k_2\frac{\beta}{\alpha}}\right)+\beta\alpha k_2e^{k_2\frac{\beta}{\alpha}}\right\}\left(B_{ji}^tb_t-b_{j;i}\right)y^j=0. \tag{28}$$

Using equations (27) and (28), we obtain

$$B^t_{ji}y^j y_t = 0 \text{ and } \left\{\alpha^2\left(k_1 e^{k_1\frac{\beta}{\alpha}} + e^{k_2\frac{\beta}{\alpha}}\right) + \beta\alpha k_2 e^{k_2\frac{\beta}{\alpha}}\right\}\left(B^t_{ji}b_t - b_{j;i}\right)y^j = 0,$$

which gives $B^t_{ji}a_{th} + B^t_{hi}a_{tj} = 0, B^t_{ji}b_t - b_{j;i} = 0$. Thus, by the well known Christoffel process we find $B^t_{ji} = 0$. Therefore we have:

Theorem 4.1. The Randers change of the generalized exponential metric (9) provides a Berwald metric if and only if $b_{j;i} = 0$, and then the Berwald connection is Riemannian $\left(\gamma^i_{jk}, \gamma^i_{0j}, 0\right)$.

Now, we consider the condition for a Finsler space F^n with the(α,β)-metric (9) to be a Douglas space. Substituting equation (25) intoequation (2.7), we get

$$\begin{aligned}
&2(\alpha - k_1\beta)e^{k_1\frac{\beta}{\alpha}}[\alpha^2 - k_1\alpha\beta + (b^2\alpha^2 - \beta^2){k_1}^2]B^{ij} \\
&-2\alpha\left[\alpha k_1 e^{k_1\frac{\beta}{\alpha}} + (\alpha + \beta k_2)e^{k_2\frac{\beta}{\alpha}}\right]\left[\alpha^2 - k_1\alpha\beta + (b^2\alpha^2 - \beta^2){k_1}^2\right]\left(s^i_0 y^j - s^j_0 y^i\right) \\
&-k_1\alpha^2\left\{(\alpha - k_1\beta)e^{k_1\frac{\beta}{\alpha}}r_{00} - 2\alpha s_0\left[\alpha k_1 e^{k_1\frac{\beta}{\alpha}} + (\alpha + \beta k_2)e^{k_2\frac{\beta}{\alpha}}\right]\right\}\left(b^i y^j - b^j y^i\right) = 0.
\end{aligned} \tag{29}$$

Let F^n be a Douglas space, i.e.,$B^{ij}$are $hp(3)$. Separating equation (29) in rational and irrational terms of y^i, because α is irrational in $\left(y^i\right)$, we find

$$\begin{aligned}
&\Big[\left(-4k_1\alpha^2\beta + 2{k_1}^3\beta^3 - 2{k_1}^3 b^2\alpha^2\beta\right)e^{k_1\frac{\beta}{\alpha}}B^{ij} \\
&+\left(\alpha k_1 e^{k_1\frac{\beta}{\alpha}} + \alpha e^{k_2\frac{\beta}{\alpha}} + \beta k_2 e^{k_2\frac{\beta}{\alpha}}\right)\left(-2\alpha^3 - 2{k_1}^2 b^2\alpha^3 + 2{k_1}^2\alpha\beta^2\right)\left(s^i_0 y^j - s^j_0 y^i\right) \\
&+{k_1}^2\alpha^2\beta e^{k_1\frac{\beta}{\alpha}}r_{00}\left(b^i y^j - b^j y^i\right) + 2k_1\alpha^3 s_0\left(\alpha k_1 e^{k_1\frac{\beta}{\alpha}} + \alpha e^{k_2\frac{\beta}{\alpha}} + \beta k_2 e^{k_2\frac{\beta}{\alpha}}\right)\left(b^i y^j - b^j y^i\right)\Big] \\
&+\alpha\Big[e^{k_1\frac{\beta}{\alpha}}B^{ij}\left(2\alpha^2 + 2{k_1}^2 b^2\alpha^2\right) + 2k_1\alpha\beta\left(\alpha k_1 e^{k_1\frac{\beta}{\alpha}} + \alpha e^{k_2\frac{\beta}{\alpha}} + \beta k_2 e^{k_2\frac{\beta}{\alpha}}\right)\left(s^i_0 y^j - s^j_0 y^i\right) \\
&-k_1\alpha^2 e^{k_1\frac{\beta}{\alpha}}r_{00}\left(b^i y^j - b^j y^i\right)\Big] = 0.
\end{aligned} \tag{30}$$

Hence equation (30) is divided into the following two equations:

$$\begin{aligned}
&\Big[\left(-4k_1\alpha^2\beta + 2{k_1}^3\beta^3 - 2{k_1}^3 b^2\alpha^2\beta\right)e^{k_1\frac{\beta}{\alpha}}B^{ij} \\
&+\left(\alpha k_1 e^{k_1\frac{\beta}{\alpha}} + \alpha e^{k_2\frac{\beta}{\alpha}} + \beta k_2 e^{k_2\frac{\beta}{\alpha}}\right)\left(-2\alpha^3 - 2{k_1}^2 b^2\alpha^3 + 2{k_1}^2\alpha\beta^2\right)\left(s^i_0 y^j - s^j_0 y^i\right) \\
&+{k_1}^2\alpha^2\beta e^{k_1\frac{\beta}{\alpha}}r_{00}\left(b^i y^j - b^j y^i\right) + 2k_1\alpha^3 s_0\left(\alpha k_1 e^{k_1\frac{\beta}{\alpha}} + \alpha e^{k_2\frac{\beta}{\alpha}} + \beta k_2 e^{k_2\frac{\beta}{\alpha}}\right)\left(b^i y^j - b^j y^i\right)\Big] = 0,
\end{aligned} \tag{31}$$

and

$$e^{k_1\frac{\beta}{\alpha}}B^{ij}\left(2\alpha^2+2{k_1}^2b^2\alpha^2\right)+2k_1\alpha\beta\left(\alpha k_1e^{k_1\frac{\beta}{\alpha}}+\alpha e^{k_2\frac{\beta}{\alpha}}+\beta k_2e^{k_2\frac{\beta}{\alpha}}\right)\left(s_0^iy^j-s_0^jy^i\right)-k_1\alpha^2e^{k_1\frac{\beta}{\alpha}}r_{00}\left(b^iy^j-b^jy^i\right)=0. \quad (32)$$

Eliminating B^{ij} from equations (31) and (32), we obtain

$$A\left(s_0^iy^j-s_0^jy^i\right)+B\left(b^iy^j-b^jy^i\right)=0, \quad (33)$$

where

$$A=2\alpha\left(-2\alpha^4-4{k_1}^2b^2\alpha^4+6{k_1}^2\alpha^2\beta^2-2{k_1}^4b^4\alpha^4+4{k_1}^4b^2\alpha^2\beta^2-2{k_1}^4\beta^4\right)\times\left(\alpha k_1e^{k_1\frac{\beta}{\alpha}}+\alpha e^{k_2\frac{\beta}{\alpha}}+\beta k_2e^{k_2\frac{\beta}{\alpha}}\right), \quad (34)$$

$$\mathrm{B}=2\alpha[{k_1}^2\alpha e^{k_1\frac{\beta}{\alpha}}r_{00}\left(\beta^3{k_1}^2-\alpha^2\beta\right)+2k_1\alpha^4\left(1+{k_1}^2b^2\right)\left(\alpha k_1e^{k_1\frac{\beta}{\alpha}}+\alpha e^{k_2\frac{\beta}{\alpha}}+\beta k_2e^{k_2\frac{\beta}{\alpha}}\right)s_0. \quad (35)$$

Transvecting (33) by b_iy_j, we have

$$As_0\alpha^2+B(b^2\alpha^2-\beta^2)=0. \quad (36)$$

The term of the equation (36) which does not contain α^2 is $-\beta^5r_{00}$. Hence there exists $hp(5)$: v_5 such that

$$-{k_1}^4\beta^5r_{00}=\alpha^2V_5. \quad (37)$$

Now, we consider the following two cases:
(i) $V_5=0$,
(ii) $V_5\neq 0, \alpha^2\not\equiv 0(\mathrm{mod}\beta)$.
Case (i). When $V_5=0$, this leads to $r_{00}=0$. Therefore, substituting $r_{00}=0$ in equation (36), we find

$$s_0(A+\gamma^2B_1)=0, \quad (38)$$

where

$$B_1=2k_1\alpha^2\left(1+{k_1}^2b^2\right)\left(\alpha k_1e^{k_1\frac{\beta}{\alpha}}+\alpha e^{k_2\frac{\beta}{\alpha}}+\beta k_2e^{k_2\frac{\beta}{\alpha}}\right).$$

If $A+\gamma^2B_1=0$, then the term $A+\gamma^2B_1=0$ which does not contain α^2 is $-4\beta^4$. Thus there exists $hp(2)$: V_2 such that

$$-{k_1}^4\beta^4=\alpha^2V_2.$$

Hence we have $V_2=0$, which leads to a contradiction. Therefore, $A+\gamma^2B_1\neq 0$. Hence, we get $s_0=0$ from (38). Substituting $s_0=0$ and $r_{00}=0$ into (33), we obtain

$$A\left(s_0^iy^j-s_0^jy^i\right)=0. \quad (39)$$

If $A=0$, then from equation (34), we obtain

$$2\alpha\left(-2\alpha^4-4{k_1}^2b^2\alpha^4+6{k_1}^2\alpha^2\beta^2-2{k_1}^4b^4\alpha^4+4{k_1}^4b^2\alpha^2\beta^2-2{k_1}^4\beta^4\right)\times$$

$$\left(\alpha k_1 e^{k_1\frac{\beta}{\alpha}}+\alpha e^{k_2\frac{\beta}{\alpha}}+\beta k_2 e^{k_2\frac{\beta}{\alpha}}\right)=0,$$

or

$$\left(-2\alpha^4-4{k_1}^2b^2\alpha^4+6{k_1}^2\alpha^2\beta^2-2{k_1}^4b^4\alpha^4+4{k_1}^4b^2\alpha^2\beta^2-2{k_1}^4\beta^4\right)\times \left(\alpha k_1 e^{k_1\frac{\beta}{\alpha}}+\alpha e^{k_2\frac{\beta}{\alpha}}+\beta k_2 e^{k_2\frac{\beta}{\alpha}}\right)=0. \quad (40)$$

The term of the equation (40) which does not contain α^2 is $-4\beta^4$. Thus, there exists $hp(2){:}\,V_2$ such that

$$-2{k_1}^4\beta^4=\alpha^2V_2,$$

from which we get $V_2=0$. This is a contradiction, therefore, $A\neq 0$. Hence, from equation (39) we get $s_0^iy^j-s_0^jy^i=0$. Now, transvecting the above equation by $y_j$we get $s_0^i=0$, which implies $s_{ij}=0$. Consequently, we find $r_{ij}=s_{ij}=0$, i.e., $b_{i;j}=0$.

Case (ii). The equation (37) shows that there exists a function $k=k(x)$ such that $r_{00}=k(x)\alpha^2$.

Thus, equation (36) which does not contain α^2 is included in the term $-\beta^5r_{00}$. Hence we get $r_{00}=0$. From equation (39), we get $A\left(s_0^iy^j-s_0^jy^i\right)=0$. If $A=0$, then it is a contradiction. Hence $A\neq 0$. Therefore, we obtain $s_0^iy^j-s_0^jy^i=0$. Again, transvecting this equation by y_j we get $s_0^i=0$. Hence, in both the cases (i) and (ii) we have $r_{ij}=0$ and $s_{ij}=0$, i.e., $b_{i;j}=0$. Conversely if $b_{i;j}=0$, then F^n is a Berwald space, so F^n is a Douglas space. Thus we have the following:

Theorem 4.2. The_Randers_change of the generalized exponential metric (9) is of the Douglas type if and only if $\alpha^2\not\equiv 0(\mathrm{mod}\beta)$ and $b_{i;j}=0$.

From Theorem 4.1 and Theorem 4.2, we have

Theorem 4.3. If the Randers change inthe generalized exponential metric (9) is of the Douglas type, then it is Berwaldian.

4. CONCLUSION

In this way we obtain the results of Randers change of the generalized Exponential metric in terms of constants k_1 and k_2. By putting the different values for $k_1$and $k_2$we can find different results for their respective values.

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