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The Locating Chromatic Number for Amalgamation of Some Complete Graphs

Amanah Yulianti, Asmiati*, Nur Hamzah, and Notiragayu
Department of Mathematics, Faculty of Mathematics and Natural Sciences
University Lampung, Bandar Lampung, Indonesia
*email: asmiati.1976@fmipa.unila.ac.id

Abstract

The locating chromatic number of a graph is a combination of partition dimension and vertex coloring, where every two adjacent vertices are in different color classes, and all vertices have a unique color code. The amalgamation of $a \geq 2$ complete graphs (K_n, $n \geq 3$) denoted by aK_n is obtained by identifying one vertex from each complete graph. In this paper, we present a novel study, a topic that has not been extensively explored in previous research, on the locating chromatic number for the amalgamation of complete graph aK_n for $2 \leq a \leq 6$ and $n \geq 3$.
Keywords: Locating chromatic number; Partition dimension; Vertex coloring; Color code; Amalgamation of complete graph.

Abstrak

Bilangan kromatik lokasi graf merupakan penggabungan dari dimensi partisi dan pewarnaan titik, yang mana setiap dua titik bertetangga berada dalam kelas warna yang berbeda dan semua titik mempunyai kode warna yang unik. Amalgamasi dari $a \geq 2$ *buah graf lengkap* (K_n, $n \geq 3$) *dinotasikan dengan* aK_n *diperoleh dengan cara menyatukan satu titik dari setiap graf lengkap* K_n. *Pada paper ini didiskusikan hasil yang belum ada sebelumnya, yaitu bilangan kromatik lokasi amalgamasi graf lengkap* aK_n *untuk* $2 \leq a \leq 6$ *dan* $n \geq 3$.
Kata Kunci: *Bilangan kromatik lokasi, Dimensi partisi, Pewarnaan titik, Kode warna, Amalgamasi graf lengkap.*

2020MSC: 05C12, 05C15

1. INTRODUCTION

The study of locating chromatic numbers is a fascinating subject in graph theory. This concept evolves from the ideas of partition dimensions and vertex coloring. The partition dimension of a graph is an extension of the metric dimension, which was first explored by Slater in 1975 [1]. The metric dimension of a graph is defined as the minimum number of members in the set representing a vertex's distances [2].

The locating chromatic number was introduced by Chartrand et al. in 2002 [3]. Let $G = (V, E)$ be a connected graph, and q is a vertex coloring in G using the l colors of G, where $q(u) \neq q(v)$ for any two adjacent vertices u, v in G. Suppose $\Pi = \{Q_1, Q_2, \dots, Q_l\}$ is a partition of $V(G)$ which is induced by coloring q. The color code $q_\Pi(w)$ of w is l-pairs tuple $(d(w, Q_1), d(w, Q_2), \dots d(w, Q_l))$ with $d(w, Q_i) = \min \{d(w,x) | x \in Q_i$ for $1 \leq i \leq l$. If all vertices in $V(G)$ have different color codes, then q is called l-locating coloring of G.The locating chromatic number of a graph G, denoted by $\chi_L(G)$ is the smallest l such that G has a l-locating coloring. Chartrand et al. [3] have obtained locating chromatic numbers from several graphs, including paths, cycles, and

* Corresponding author
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double star graphs. One year later, Chartrand et al. [4] obtained the locating chromatic number of some graphs with locating chromatic number $(n-1)$.

Theorem 1.1. (see[3]). *The locating chromatic number of a complete graph* (K_n) *is* n *for* $n \geq 2$.

Asmiati et al. [5] determined the locating chromatic number for the amalgamation of stars. In 2012, Asmiati found the locating chromatic number for firecracker graphs [6] and characterized graphs containing cycles with a locating chromatic number of three [7]. Baskoro and Asmiati [8] later characterized all trees with a locating chromatic number of three. Asmiati also obtained the locating chromatic number for nonhomogeneous caterpillars and firecracker graphs [9], certain barbell graphs [10], specific operations on origami graphs [11], barbell shadow path graphs [12], and the disjoint union of some double star graphs [13]. Irawan determined the locating chromatic number for subdivision firecracker graphs [14], various operations on generalized Petersen graphs [15][16], origami graphs [17] and provided a procedure for finding it [18], as well as the upper bounds of the locating chromatic numbers for shadow cycle graphs [19]. Behtoei found the locating chromatic number for Kneser graphs [20] and the Cartesian product of graphs [21]. Ghanem et al. [22] obtained the locating chromatic number for powers of paths and cycles. Furuya and Matsumoto [23] established upper bounds on the locating chromatic number for trees. Prawinasti et al. [24] determined the locating chromatic number for the split graph of the cycle. Damayanti et al. [25] found the locating chromatic number for some modified paths with cycles having a locating chromatic number of four. Finally, Rahmatalia et al. [26] determined the locating chromatic number for split graphs of paths.

Based on the literature study above, there is still no general formula for finding the locating chromatic number for any graph. So, we will discuss the locating chromatic number for amalgamation of complete graph and its barbell. The amalgamation of $a \geq 2$ complete graphs ($K_n, n \geq 3$) denoted by aK_n is obtained by identifying one vertex from each complete graph. We call the identified vertex as the center (denoted by p). Let $V(aK_n) = \{ p, v_i^j \,|\, i = 1, 2, \dots, n-1;\ j = 1,2,3, \dots, a\,;\ a \geq 2 \}$. An example for amalgamation of complete graph $4K_4$ with vertices label can be showed in Figure 1.

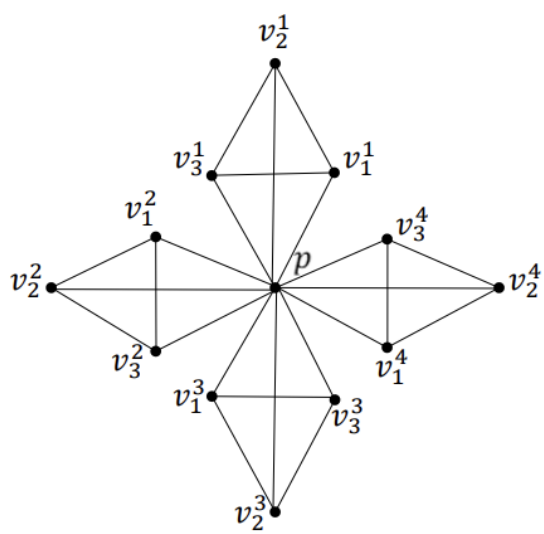

Figure 1. Amalgamation of complete graph $4K_4$ with vertices label.

2. METHODS

To obtain the locating chromatic number of amalgamation of complete graph, we determine the lower and upper bound with construct the vertex coloring by looking at the graph structure. Vertex coloring starts with the smallest label so that a minimum vertex coloring is obtained that meets the location coloring requirements. If the lower bound and upper bound of the locating chromatic number of amalgamation of complete graph (aK_n) are identical, for example x, then the locating chromatic number is $\chi_L(aK_n) = x$.

3. RESULTS AND DISCUSSION

Theorem 3.1 $\chi_L(aK_n) = n + 1$ *for* $n \geq 3$ *and* $a \in \{2,3\}$.

Proof.

First, we determine the lower bound of the locating chromatic number of the graph aK_n for $n \geq 3$ and $a \in \{2,3\}$. Amalgamation of complete graph aK_n contains K_n, by Theorem 1.1, we have $\chi_L(aK_n) \geq n$. Suppose that there is a n-locating coloring of aK_n for $n \geq 3$ and $a \in \{2,3\}$. Since aK_n containing at least two K_n, then there are two vertices have the same color codes, a contrary. So, $\chi_L(aK_n) \geq n + 1$.

Next, we determined the upper bound of the locating chromatic number of aK_n. Let q be a $(n + 1)$-coloring in aK_n for $n \geq 3$ and $a \in \{2,3\}$. as follows: contains at least two K_n.

$q(p) = 1.$

$$q(v_i^j) = \begin{cases} 2; \; i = 1, & j = 3. \\ i + 1; 1 \leq i \leq n - 1, & j = 1. \\ i + 2; 1 \leq i \leq n - 1, & j = 2; \\ 2 \leq i \leq n - 1, & j = 3. \end{cases}$$

Then the code of vertices are:

$$q_\Pi(p) = \begin{cases} 0 & , 1^{st} \text{ tuple.} \\ 1 & , \text{tuple other.} \end{cases}$$

$$q_\Pi(v_i^j) = \begin{cases} 0 & , 2^{nd} \text{ tuple; } i = 1 \text{ and } j = 3; \\ & i + 1^{th} \text{ tuple; } 1 \leq i \leq n - 1 \text{ and } j = 1; \\ & i + 2^{th} \text{tuple; } 1 \leq i \leq n - 1 \text{ and } j = 2; \\ & i + 2^{th} \text{tuple; } 2 \leq i \leq n - 1 \text{ and } j = 3. \\ 2 & , 2^{nd} \text{ tuple; } 1 \leq i \leq n - 1 \text{ and } j = 2; \\ & 3^{rd} \text{ tuple; } 1 \leq i \leq n - 1 \text{ and } j = 3; \\ & n + 1^{th} \text{ tuple; } 1 \leq i \leq n - 1 \text{ and } j = 1. \\ 1 & , \text{otherwise.} \end{cases}$$

Since all vertices in aK_n for $n \geq 3$ and $a \in \{2,3\}$ have distinct color codes, then q is a locating coloring. So, $\chi_L(aK_n) \leq n+1$. Therefore $\chi_L(aK_n) = n+1$. ■

Theorem 3.2. *The locating chromatic number for amalgamation of complete graph* $(4K_n)$,

$$\chi_L(4K_n) = \begin{cases} 5; & if\ n = 3, \\ n+1; & if\ n \geq 4. \end{cases}$$

Proof.
Consider two cases.

Case 1 ($n = 3$).
The graph $4K_3$ contains K_3, based on Theorem 1.1, we have $\chi_L(4K_3) \geq 3$. Suppose that there is a 3-locating coloring of $4K_3$. Without loss of generality, we assign $q(p) = 1$, color 2 or 3 to other vertices. It is clearly, there are two vertices have the same color codes. So, $\chi_L(4K_3) \geq 4$. Suppose that there is a 4 locating of $4K_3$. Without loss of generality, we assign $q(p) = 1$, color $2, 3$ or 4 to other vertices. Then there are two pairs vertices with the same color such that have the same color codes, a contrary. As a result, $\chi_L(4K_3) \geq 5$.

Next, vertex coloring is given to determine the locating chromatic number of the graph. Let q be a vertex coloring that uses 5 colors. The $4K_3$ graph is given a color class such that it is obtained

$$Q_1 = \{p\}, Q_2 = \{v_1^1, v_1^3\}, Q_3 = \{v_2^1, v_1^2, v_1^4\}, Q_4 = \{v_2^2, v_2^3\}, Q_5 = \{v_2^4\}.$$

Then, the color codes of vertices are:

$$q_\pi(p) = \{0,1,1,1,1\}; \qquad q_\pi(v_1^1) = \{1,0,1,2,2\}; \qquad q_\pi(v_2^1) = \{1,1,0,2,2\};$$

$$q_\pi(v_1^2) = \{1,2,0,1,2\}; \qquad q_\pi(v_2^2) = \{1,2,1,0,2\}; \qquad q_\pi(v_1^3) = \{1,0,2,1,2\};$$

$$q_\pi(v_2^3) = \{1,1,2,0,2\}; \qquad q_\pi(v_1^4) = \{1,2,0,2,1\}; \qquad q_\pi(v_2^4) = \{1,2,1,2,0\}.$$

Thus, since all vertices in $4K_3$ have distinct color codes, then q is a 5-locating coloring. So, $\chi_L(4K_3) \geq 5$. Therefore, we have $\chi_L(4K_3) = 5$.

Case 2 ($n \geq 4$).
We will first determine the lower bound of the locating chromatic number of the graph $4K_n$, Since $4K_n$ contains at least two K_n and based on Theorem 1.1, then $\chi_L(aK_n) \geq n+1$. Consequently, $\chi_L(4K_n) \geq n+1$.

Next, to determine the upper bound of the locating chromatic number of $4K_n$ for $n \geq 4$. Let q be a coloring using $(n+1)$ colors as follows:

$q(p) = 1$.

$$q(v_i^j) = \begin{cases} 2; \ i = 1, j = 3. \\ i+1; 1 \le i \le n-1, \quad j = 1; \\ \quad i = 1,2, \quad j = 4. \\ i+2; 1 \le i \le n-1, \quad j = 2; \\ \quad 2 \le i \le n-1, \quad j = 3; \\ \quad 3 \le i \le n-1, \quad j = 4. \end{cases}$$

Then, the color code of vertices are:

$$q_\Pi(p) = \begin{cases} 0 & , 1^{st} \text{ tuple} \\ 1 & , \text{tuple other} \end{cases}.$$

$$q_\Pi(v_i^j) = \begin{cases} 0 & , 2^{nd} \text{ tuple}; \ i = 1 \text{ and } j = 3; \\ & i+1^{th} \text{ tuple}; \ 1 \le i \le n-1 \text{ and } j = 1; \\ & i+1^{th} \text{ tuple}; \ i = 1,2 \text{ and } j = 4; \\ & i+2^{th} \text{ tuple}; \ 1 \le i \le n-1 \text{ and } j = 2; \\ & i+2^{th} \text{ tuple}; \ 2 \le i \le n-1 \text{ and } j = 3; \\ & i+2^{th} \text{ tuple}; \ 3 \le i \le n-1 \text{ and } j = 4. \\ 2 & , 2^{nd} \text{ tuple}; \ 1 \le i \le n-1 \text{ and } j = 2; \\ & 3^{rd} \text{ tuple}; \ 1 \le i \le n-1 \text{ and } j = 3; \\ & 4^{th} \text{ tuple}; \ 1 \le i \le n-1 \text{ and } j = 4; \\ & n+1^{th} \text{ tuple}; 1 \le i \le n-1 \text{ and } j = 1. \\ 1 & , \text{otherwise}. \end{cases}$$

Since all vertices in $4K_n$ for $n \ge 4$ have distinct color codes, then q is a locating coloring. So, $\chi_L(4K_n) \le n+1$. Therefore $\chi_L(4K_n) = n+1$. ■

Theorem 3.3. *The locating chromatic number of amalgamation of complete graph* $(5K_n)$,

$$\chi_L(5K_n) = \begin{cases} n+2; \ if \ \ n = 3,4, \\ n+1; \ if \ n \ge 5\,. \end{cases}$$

Proof.
Consider two cases.

Case 1. Consider the following two subcases.
Subcase 1.1 ($n = 3$)

We will first determine the lower bound of the locating chromatic number of the graph $5K_3$ because the graph $5K_3$ contains K_3; based on Theorem 1.1, we obtain $\chi_L(5K_3) \ge 3$. Suppose that there is a $3-$locating coloring of $4K_3$. Without loss of generality, we assign $q(p) = 1$, then the vertices neighboring p are given the colors 2 or 3. It is clearly that there are two vertices have the same color code, a contrary. So, $\chi_L(5K_3) \ge 4$. Similarly, if we use 4 colors, as a result, we have $\chi_L(5K_3) \ge 5$.

Next, vertex coloring is given to determine the locating chromatic number of the graph. Let q be a vertex coloring that uses 5 colors. The $5K_3$ graph is given a color class such that it is obtained

$$Q_1 = \{p\}, Q_2 = \{v_1^1, v_1^3\}, Q_3 = \{v_2^1, v_1^2, v_1^4\}, Q_4 = \{v_2^2, v_2^3, v_1^5\}, Q_5 = \{v_2^4, v_2^5\}.$$

Then, the color codes of vertices are:

$q_\pi(p) = \{0,1,1,1,1,\}$; $q_\pi(v_1^1) = \{1,0,1,2,2\}$; $q_\pi(v_2^1) = \{1,1,0,2,2\}$;

$q_\pi(v_1^2) = \{1,2,0,1,2\}$; $q_\pi(v_2^2) = \{1,2,1,0,2\}$; $q_\pi(v_1^3) = \{1,0,2,1,2\}$;

$q_\pi(v_2^3) = \{1,1,2,0,2\}$; $q_\pi(v_1^4) = \{1,2,0,2,1\}$; $q_\pi(v_2^4) = \{1,2,1,2,0\}$

$q_\pi(v_1^5) = \{1,2,2,0,1\}$; $q_\pi(v_2^5) = \{1,2,2,1,0\}$.

Thus, since all vertices in $5K_3$ have distinct color codes, then q is a 5-locating coloring. So, $\chi_L(5K_3) \geq 5$. Therefore, we have $\chi_L(5K_3) = 5$.

Subcase 1.2 ($n = 4$)

We will first determine the lower bound of the locating chromatic number of the graph $5K_4$. Since the graph $5K_4$ contains K_4, based on Theorem 1.1, we obtain $\chi_L(5K_4) \geq 4$. Suppose that there is a $4 -$ locating coloring of $5K_4$. Without loss of generality, we assign $q(p) = 1$, then the vertices neighboring p are given the colors 2,3 or 4. It is clear that there are two vertices that have the same color codes, a contrary. So, $\chi_L(5K_4) \geq 5$. Similarly, if we use 5 colors. As a results, we have $\chi_L(5K_4) \geq 6$.

Let q be a vertex coloring that uses 6 colors. The $5K_4$ graph is given a color class such that it is obtained

$$Q_1 = \{p\}, Q_2 = \{v_1^1, v_1^3, v_1^4\}, Q_3 = \{v_2^1, v_1^2, v_2^4, v_1^5\}, Q_4 = \{v_3^1, v_2^2, v_2^3, v_2^5\},$$

$$Q_5 = \{v_3^2, v_3^3, v_3^4\}, Q_6 = \{v_3^5\}.$$

Then, the color codes of vertices are:

$q_\pi(p) = \{0,1,1,1,1,1\}$; $q_\pi(v_1^1) = \{1,0,1,1,2,2\}$; $q_\pi(v_2^1) = \{1,1,0,1,2,2\}$;

$q_\pi(v_3^1) = \{1,1,1,0,2,2\}$; $q_\pi(v_1^2) = \{1,2,0,1,1,2\}$; $q_\pi(v_2^2) = \{1,2,1,0,1,2\}$;

$q_\pi(v_3^2) = \{1,2,1,1,0,2\}$; $q_\pi(v_1^3) = \{1,0,2,1,1,2\}$; $q_\pi(v_2^3) = \{1,1,2,0,1,2\}$;

$q_\pi(v_3^3) = \{1,1,2,1,0,2\}$; $q_\pi(v_1^4) = \{1,0,1,2,1,2\}$; $q_\pi(v_2^4) = \{1,1,0,2,1,2\}$;

$q_\pi(v_3^4) = \{1,1,1,2,0,2\}$.

Thus, since all vertices in $5K_4$ have distinct color code, then q is a 6-locating coloring. So, $\chi_L(5K_4) \geq 6$. Therefore, we have $\chi_L(5K_4) = 6$.

Case 2 ($n \geq 5$).

We will first determine the lower bound of the locating chromatic number of the graph $5K_n$. Since $5K_n$ contains at least two K_n and based on Theorem 1.1, then $\chi_L(5K_n) \geq n+1$.

Next, to determine the upper bound of the locating chromatic number of $5K_n$ for $n \geq 5$. Let q be a coloring using $(n+1)$ colors as follows:

$q(p) = 1$.

$$q(v_i^j) = \begin{cases} 2; \ i = 1, j = 3 \\ i+1; 1 \leq i \leq n-1, \quad j = 1 \\ \quad i = 1,2,3, \quad j = 5 \\ i+2; 1 \leq i \leq n-1, \quad j = 2 \\ \quad 2 \leq i \leq n-1, \quad j = 3 \\ \quad 4 \leq i \leq n-1, \quad j = 5 \end{cases}$$

Then, the color codes of vertices are:

$$q_\Pi(p) = \begin{cases} 0 & , 1^{st} \text{ tuple} \\ 1 & , \text{tuple other} \end{cases}$$

$$q_\Pi(v_i^j) = \begin{cases} 0 & , 2^{nd} \text{ tuple; } i = 1 \text{ and } j = 3; \\ & i+1^{th} \text{ tuple; } 1 \leq i \leq n-1 \text{ and } j = 1; \\ & i+1^{th} \text{ tuple; } i = 1,2 \text{ and } j = 4; \\ & i+1^{th} \text{ tuple; } i = 1,2,3 \text{ and } j = 5; \\ & i+2^{th} \text{ tuple; } 1 \leq i \leq n-1 \text{ and } j = 2; \\ & i+2^{th} \text{ tuple; } 2 \leq i \leq n-1 \text{ and } j = 3; \\ & i+2^{th} \text{ tuple; } 3 \leq i \leq n-1 \text{ and } j = 4; \\ & n+1^{th} \text{ tuple; } i = 4 \text{ and } j = 5. \\ 2 & , 2^{nd} \text{ tuple; } 1 \leq i \leq n-1 \text{ and } j = 2; \\ & 3^{rd} \text{ tuple; } 1 \leq i \leq n-1 \text{ and } j = 3; \\ & 4^{th} \text{ tuple; } 1 \leq i \leq n-1 \text{ and } j = 4; \\ & 5^{th} \text{ tuple; } 1 \leq i \leq n-1 \text{ and } j = 5; \\ & n+1^{th} \text{ tuple; } 1 \leq i \leq n-1 \text{ and } j = 1. \\ 1 & , \text{otherwise.} \end{cases}$$

Since all vertices in $5K_n$ for $n \geq 5$ have distinct color codes, then q is a locating coloring. So, $\chi_L(5K_n) \leq n+1$. Therefore $\chi_L(5K_n) = n+1$. ■

Theorem 3.4. *The locating chromatic number of amalgamation of complete graph* $6K_n$,

$$\chi_L(6K_n) = \begin{cases} n+2; \text{if } n = 3,4,5 \\ n+1; \text{if } n \geq 6. \end{cases}$$

Proof.
Consider two cases.

Case 1. Consider the following three subcases.
Subcase 1.1 ($n = 3$)

We will first determine the lower bound of the locating chromatic number of the graph 6. Since the graph $6K_3$ contains K_3, based on Theorem 1.1, we obtain $\chi_L(6K_3) \geq 3$. Suppose that there is a $3-$locating coloring of $6K_3$. Without loss of generality, we assign $q(p) = 1$, then the vertices neighboring p are given the colors 2 or 3. It is clear that two vertices that have the same color codes, a contrary. So, $\chi_L(6K_3) \geq 4$. Similarly, if we use 5 colors, as a result, we have $\chi_L(6K_3) \geq 5$.

Next, vertex coloring is given to determine the locating chromatic number of the graph. Let q be a vertex coloring that uses 5 colors. The $6K_3$ graph is given a color class such that it is obtained

$$Q_1 = \{p\}, Q_2 = \{v_1^1, v_1^3, v_1^5\}, Q_3 = \{v_2^1, v_1^2, v_1^4\}, Q_4 = \{v_2^2, v_2^3, v_2^6\}, Q_5 = \{v_2^4, v_2^5, v_2^6\}.$$

Then, the color codes of vertices are:

$q_\pi(p) = \{0,1,1,1,1\}$; $q_\pi(v_1^1) = \{1,0,1,2,2\}$; $q_\pi(v_2^1) = \{1,1,0,2,2\}$;

$q_\pi(v_1^2) = \{1,2,0,1,2\}$; $q_\pi(v_2^2) = \{1,2,1,0,2\}$; $q_\pi(v_1^3) = \{1,0,2,1,2\}$;

$q_\pi(v_2^3) = \{1,1,2,0,2\}$; $q_\pi(v_1^4) = \{1,2,0,2,1\}$; $q_\pi(v_2^4) = \{1,2,1,2,0\}$;

$q_\pi(v_1^5) = \{1,0,2,2,1\}$; $q_\pi(v_2^5) = \{1,1,2,2,0\}$; $q_\pi(v_1^6) = \{1,2,2,0,1\}$;

$q_\pi(v_2^6) = \{1,2,2,1,0\}$.

Thus, since all vertices in $6K_3$ have distinct color code, then q is a 5-locating coloring. So, $\chi_L(6K_3) \geq 5$. Therefore, we have $\chi_L(6K_3) = 5$.

Subcase 1.2 ($n = 4$)

We will first determine the lower bound of the locating chromatic number of the graph $6K_4$. Since the graph $6K_4$ contains K_4, based on Theorem 1.1, we obtain $\chi_L(6K_4) \geq 4$. Without loss of generality, we assign $q(p) = 1$, and then the vertices neighboring p are given the colors 2,3,3 or 4. It is clear that two vertices have the same color codes, a contrary. So, $\chi_L(6K_4) \geq 5$. Similarly, if we use 5 color, subsequentlyy, $\chi_L(6K_4) \geq 6$.

Next, vertex coloring is given to determine the locating chromatic number of the graph. Let q be a vertex coloring that uses 6 colors. The $6K_4$ graph is given a color class such that it is obtained

$$Q_1 = \{p\}, Q_2 = \{v_1^1, v_1^3, v_1^4\}, Q_3 = \{v_2^1, v_1^2, v_2^4, v_1^5, v_1^6\}, Q_4 = \{v_3^1, v_2^2, v_2^3, v_2^5\},$$

$$Q_5 = \{v_3^2, v_3^3, v_3^4, v_2^6\}, Q_6 = \{v_3^5, v_3^6\}.$$

Then, the color codes of vertices are:

$q_\pi(p) = \{0,1,1,1,1,1\}$; $q_\pi(v_1^1) = \{1,0,1,1,2,2\}$; $q_\pi(v_2^1) = \{1,1,0,1,2,2\}$;

$q_\pi(v_3^1) = \{1,1,1,0,2,2\}$; $q_\pi(v_1^2) = \{1,2,0,1,1,2\}$; $q_\pi(v_2^2) = \{1,2,1,0,1,2\}$;

$q_\pi(v_3^2) = \{1,2,1,1,0,2\}$; $q_\pi(v_1^3) = \{1,0,2,1,1,2\}$; $q_\pi(v_2^3) = \{1,1,2,0,1,2\}$;

$q_\pi(v_3^3) = \{1,1,2,1,0,2\}$; $q_\pi(v_1^4) = \{1,0,1,2,1,2\}$; $q_\pi(v_2^4) = \{1,1,0,2,1,2\}$;

$q_\pi(v_3^4) = \{1,1,1,2,0,2\}$; $q_\pi(v_1^5) = \{1,2,0,1,2,1\}$; $q_\pi(v_2^5) = \{1,2,1,0,2,1\}$;

$q_\pi(v_3^5) = \{1,2,1,1,2,0\}$; $q_\pi(v_1^6) = \{1,2,0,2,1,1\}$; $q_\pi(v_2^6) = \{1,2,1,2,0,1\}$;

$q_\pi(v_3^6) = \{1,2,1,2,1,0\}$.

Thus, since all vertices in $6K_4$ have distinct color codes, q is a 6-locating coloring. So, $\chi_L(6K_4) \geq 6$. Therefore, we have $\chi_L(6K_4) = 6$.

Subcase 1.3 ($n = 5$)

We will first determine the lower bound of the locating chromatic number of the graph $6K_5$. Since the graph $6K_5$ contains K_5, based on Theorem 1.1, we obtain $\chi_L(6K_5) \geq 5$. Without loss of generality, we assign $q(p) = 1$, and then the vertices neighboring p are given the colors 2,3,4 or 5. It is clear that two vertices have the same color codes, a contrary. So, $\chi_L(6K_5\ ) \geq 6$. Similarly, if we use 6 colors, thus, $\chi_L(6K_5) \geq 7$.

Next, vertex coloring is given to determine the locating chromatic number of the graph. Let q be a vertex coloring that uses 7 colors. The $6K_5$ graph is given a color class such that it is obtained

$$Q_1 = \{p\}, Q_2 = \{v_1^1, v_1^3, v_1^4, v_1^5\}, Q_3 = \{v_2^1, v_1^2, v_2^4, v_2^5, v_1^6\}, Q_4 = \{v_3^1, v_2^2, v_2^3, v_3^5, v_2^6\},$$

$$Q_5 = \{v_4^1, v_3^2, v_3^3, v_3^4, v_3^6\}, Q_6 = \{v_4^2, v_4^3, v_4^4, v_4^5\}, Q_7 = \{v_4^6\}.$$

Then, the color codes of vertices are:

$q_\pi(p) = \{0,1,1,1,1,1,1\}$; $q_\pi(v_1^1) = \{1,0,1,1,1,2,2\}$; $q_\pi(v_2^1) = \{1,1,0,1,1,2,2\}$;

$q_\pi(v_3^1) = \{1,1,1,0,1,2,2\}$; $q_\pi(v_4^1) = \{1,1,1,1,0,2,2\}$; $q_\pi(v_1^2) = \{1,2,0,1,1,1,2\}$;

$q_\pi(v_2^2) = \{1,2,1,0,1,1,2\}$; $q_\pi(v_3^2) = \{1,2,1,1,0,1,2\}$; $q_\pi(v_4^2) = \{1,2,1,1,1,0,2\}$;

$q_\pi(v_1^3) = \{1,0,2,1,1,1,2\}$; $q_\pi(v_2^3) = \{1,1,2,0,1,1,2\}$; $q_\pi(v_3^3) = \{1,1,2,1,0,1,2\}$;

$q_\pi(v_4^3) = \{1,1,2,1,1,0,2\}$; $q_\pi(v_1^4) = \{1,0,1,2,1,1,2\}$; $q_\pi(v_2^4) = \{1,1,0,2,1,1,2\}$;

$q_\pi(v_3^4) = \{1,1,1,2,0,1,2\}$; $q_\pi(v_4^4) = \{1,1,1,2,1,0,2\}$; $q_\pi(v_1^5) = \{1,0,1,1,2,1,2\}$;

$q_\pi(v_2^5) = \{1,1,0,1,2,1,2\}$; $q_\pi(v_3^5) = \{1,1,1,0,2,1,2\}$; $q_\pi(v_4^5) = \{1,1,1,1,2,0,2\}$;

$q_\pi(v_1^6) = \{1,2,0,1,1,2,1\}$; $q_\pi(v_2^6) = \{1,2,1,0,1,2,1\}$; $q_\pi(v_3^6) = \{1,2,1,1,0,2,1\}$;

$q_\pi(v_4^6) = \{1,2,1,1,1,2,0\}$.

Thus, since all vertices in $6K_5$ have distinct color codes, q is a 7-locating coloring. So, $\chi_L(6K_5) \geq 7$. Therefore, we have $\chi_L(6K_5) = 7$.

Case 2 ($n \geq 6$).

We will first determine the lower bound of the locating chromatic number of the graph $6K_n$. Since $6K_n$ contains at least two K_n based on Theorem 1.1, then $\chi_L(aK_n) \geq n+1$.

Next, to determine the upper bound of the locating chromatic number of $6K_n$ for $n \geq 6$. Let q be a coloring using $(n+1)$ colors as follows:

$q(p) = 1.$

$$q(v_i^j) = \begin{cases} 2; \ i = 1, j = 3 \\ 7; i = 5, \ j = 6 \\ i+1; 1 \leq i \leq n-1, \ j = 1 \\ \quad i = 1,2, \ j = 4 \\ \quad i = 1,2,3, \ j = 5 \\ \quad i = 1,2,3,4, \ j = 6 \\ i+2; 1 \leq i \leq n-1, \ j = 2 \\ \quad 2 \leq i \leq n-1, \ j = 3 \\ \quad 3 \leq i \leq n-1, \ j = 4 \\ \quad 4 \leq i \leq n-1, \ j = 5 \end{cases}$$

Then, the color code of vertices are:

$$q_\Pi(p) = \begin{cases} 0 & , 1^{st} \text{ tuple.} \\ 1 & , \text{otherwise.} \end{cases}$$

$$q_\Pi(v_i^j) = \begin{cases} 0 & , 2^{nd} \text{ tuple; } i = 1 \text{ and } j = 3; \\ & i+1^{th} \text{ tuple; } 1 \leq i \leq n-1 \text{ and } j = 1; \\ & i+1^{th} \text{ tuple; } 1 \leq i \leq n-2 \text{ and } j = 6; \\ & i+1^{th} \text{ tuple; } 1 \leq i \leq n-3 \text{ and } j = 5; \\ & i+1^{th} \text{ tuple; } 1 \leq i \leq n-4 \text{ and } j = 4; \\ & i+2^{th} \text{ tuple; } 1 \leq i \leq n-1 \text{ and } j = 2; \\ & i+2^{th} \text{ tuple; } 2 \leq i \leq n-1 \text{ and } j = 3; \\ & i+2^{th} \text{ tuple; } 3 \leq i \leq n-1 \text{ and } j = 4; \\ & i+2^{th} \text{ tuple; } 4 \leq i \leq n-1 \text{ and } j = 5; \\ & n+2^{th} \text{ tuple; } i = 5 \text{ and } j = 6. \\ 2 & , n+1^{th} \text{ tuple; } 1 \leq i \leq n-1 \text{ and } j = 1; \\ & 2^{nd} \text{ tuple; } 1 \leq i \leq n-1 \text{ and } j = 2; \\ & 3^{rd} \text{ tuple; } 1 \leq i \leq n-1 \text{ and } j = 3; \\ & 4^{th} \text{ tuple; } 1 \leq i \leq n-1 \text{ and } j = 4; \\ & 5^{th} \text{ tuple; } 1 \leq i \leq n-1 \text{ and } j = 5; \\ & 6^{th} \text{ tuple; } 1 \leq i \leq n-1 \text{ and } j = 6. \\ 1 & , \text{otherwise.} \end{cases}$$

Since all vertices in $6K_n$ for $n \geq 6$ have distinct color codes, q is a locating coloring. So, $\chi_L(6K_n) \leq n+1$. Therefore $\chi_L(6K_n) = n+1$. ∎

Figure 2 is an example of the locating chromatic number for the amalgamation of complete graph $4K_4$, with the locating chromatic number being 5.

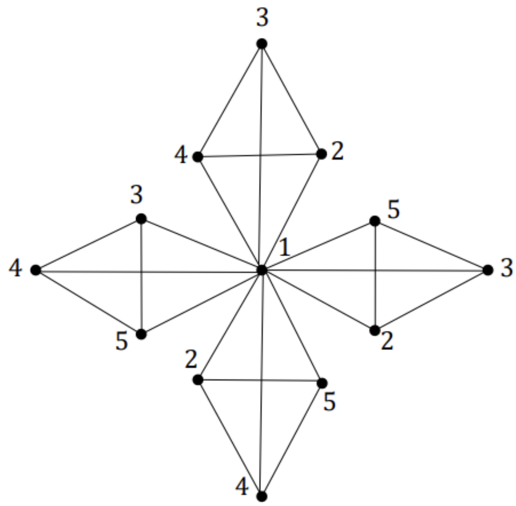

Figure 2. A minimum locating coloring of $4K_4$

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