InPrime: Indonesian Journal of Pure and Applied Mathematics
Vol. 6, No. 1 (2024), pp. 38 – 51, doi: 10.15408/inprime.v6i1.37158
p-ISSN 2686-5335, e-ISSN: 2716-2478

Bivariate Distributions and Copula-Tvar Estimates: A Comparative Study Based on The Selected Financial Returns and Marginal Distributions

Anayo Charles Iwuji[1], Ben Ifeanyichukwu Oruh[2], Joy Chioma Nwabueze[3], and Emmanuel Wilfred Okereke[4]
[1,3,4]Department of Statistics, College of Physical and Applied Sciences,
Michael Okpara University of Agriculture, Umudike, Nigeria
[2]Department of Mathematics, College of Physical and Applied Sciences,
Michael Okpara University of Agriculture, Umudike, Nigeria
Email: iwuji.charles@mouau.edu.ng

Abstract

This study explores the joint distribution of bivariate financial returns on DJIA-S&P500 and SSE-SZSE, employing copulas and model selection criteria to identify the most suitable distribution. The aim is to estimate Conditional Tail Value at Risk (C-TVaR) at various confidence levels for portfolio risk management. Unlike previous studies, which typically focus on univariate analysis, this research examines into the joint distribution of bivariate financial returns. Additionally, it introduces the application of copulas and model selection criteria to determine the optimal joint distribution for portfolio risk assessment, offering valuable insights for financial decision-makers. Several copulas and model selection criteria are employed to assess the joint distribution of bivariate financial returns. By evaluating the minimum values of model selection criteria such as Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC), the Student's t copula is identified as the most appropriate copula. C-TVaR estimates are then obtained at different confidence levels using the selected copula and various combinations of marginal distributions, namely, normal, Student's t, Cauchy, and alpha power transformed logistic (APTL) marginal distributions. Empirical results demonstrate that Student's t copula models with APTL-Student's t and APTL-APTL marginals gave the smallest expected portfolio losses for the DJIA-S&P500 and SSE-SZSE portfolios, respectively. These insights contribute to enhancing portfolio risk management strategies, particularly in assessing tail risk at different confidence levels.

Keywords: Alpha power transformed logistic distribution, bivariate copula, C-TVaR, model selection criteria, portfolio loses.

Abstrak

Studi ini mengeksplorasi distribusi bersama dari keuntungan finansial pada DJIA-S&P500 dan SSE-SZSE, dengan menggunakan kopula dan kriteria pemilihan model untuk mengidentifikasi distribusi yang paling cocok. Tujuannya adalah untuk memperkirakan Conditional Tail Value at Risk (C-TVaR) pada berbagai tingkat kepercayaan untuk manajemen risiko portofolio. Berbeda dengan penelitian sebelumnya, yang umumnya berfokus pada analisis univariat, penelitian ini menyelidiki distribusi bersama dari keuntungan finansial. Selain itu, penelitian ini memperkenalkan aplikasi kopula dan kriteria pemilihan model untuk menentukan distribusi bersama optimal untuk penilaian risiko portofolio, memberikan wawasan berharga bagi pengambil keputusan finansial. Beberapa kopula dan kriteria pemilihan model digunakan untuk menilai distribusi bersama dari keuntungan finansial. Dengan mengevaluasi nilai minimum dari kriteria pemilihan model seperti Akaike Information Criterion (AIC) dan Bayesian Information Criterion (BIC), kopula Student's t diidentifikasi sebagai kopula yang paling cocok. Estimasi C-TVaR kemudian diperoleh pada tingkat kepercayaan yang berbeda menggunakan kopula yang dipilih dan berbagai kombinasi distribusi marginal, yaitu distribusi marginal normal, Student's t, Cauchy, dan alpha power transformed logistic (APTL). Hasil empiris menunjukkan bahwa model kopula Student's t dengan distribusi marginal APTL-Student's t dan APTL-APTL memberikan ekspetasi kerugian portofolio terkecil untuk masing-masing portofolio DJIA-S&P500 dan SSE-SZSETemuan ini

* Corresponding author
Submitted January 2nd, 2024, Revised May 20th, 2024,
Accepted for publication May 22nd, 2024, Published Online May 31st, 2024

berkontribusi untuk meningkatkan strategi manajemen risiko portofolio, khususnya dalam menilai risiko ekor pada tingkat kepercayaan yang berbeda.
Kata Kunci: *distribusi* Alpha power transformed logistic*, kopula bivariat, C-TVaR, kriteria pemilihan model, kehilangan portofolio.*

2020MSC: 62H05

1. INTRODUCTION

Stock portfolio risk management is a challenge faced by financial analysts in stock portfolio investment. The well-known value-at-risk (VaR) and tail value-at-risk (TVaR) metrics are valuable risk measures stock investment analysts use. They provide the requisite information about the maximum expected loss for any investment in a stock index based on a given confidence level. Though different authors have applied the VaR and TVaR risk measures, the TVaR measure has an edge over VaR as it provides the expected maximum portfolio loss beyond VaR. These VaR and TVaR risk measures have been applied for single stock index investment and in cases of portfolio of stock investment. But financial analysts, however, encourage stock portfolio investment rather than for a single stock. This is because by spreading investment across different assets in a portfolio, the investors can diversify their investments and are less likely to have their portfolio wiped out like an adverse event would affect a single asset.

Many researchers have modeled and estimated the TVaR of stock portfolio investment. Simulated TVaR estimates based on Monte Carlo for some companies listed in the LQ45 stock index at 70%, 80%, and 90% confidence levels were proposed by [1]. Shen et al. [2] established empirical likelihood-based estimation with high-order precision for TVaR [3] and compared the performance of nonparametric estimators of TVaR for varying p. They compared the empirical estimator, kernel-based estimator, Brazauskas et al.'s estimator, tail-trimmed estimator and the filtered historical method. The nonparametric kernel methods were combined with extreme-value statistics to find the estimator for TVaR by [4]. Castaner et al. [5] estimated VaR and TVaR of insurance and financial companies using the normal power approximation. The precision of the approximations was checked with the exponential, Pareto, and lognormal distributions. The TVaR of several insurance portfolios consisting of several lines of risk linked by the FGM copula with exponential marginal was estimated by [6]. Brazauskas et al. [7] and Kaiser et al. [8] proposed point and interval estimators for TVaR and proved the consistency of the point estimator. Several extensions of TVaR have also been developed and modified by introducing a fixed boundary, instead of infinity, for values beyond the quantile (see [9], [10], and [11]). The works of [1] $-$ [11] did not apply copula in the estimation of the TVaR which will allow the investigation of tail dependencies of the data

Copula has been applied extensively in the modeling of stock market data because of its ability to combine univariate volatility models into flexible multivariate distributions of portfolio returns. It allows the investigation of tail dependencies, which is particularly interesting in financial data risk estimation. In Syuhada et al. [12], the Dependent value-at-risk measure (DTVaR) model was proposed, and parameter estimates of DTVaR using Farlie-Gumbel-Morgenstern (FGM) copula with Pareto marginal were studied. However, the DTVaR estimates with Pareto marginal were not compared with those of other probability distributions to determine the distribution that gives the least dependent tail-value-at-risk. Dutta and Suparna [13] used the Gaussian, Student's t, Clayton, Gumbel, and Frank copulas to estimate the value at risk and tail value at risk of returns of Novartis(NVS) and Pfizer(PFE) stock portfolio from the United States market using normal and Student's t distributions as marginals for the univariate stock return series. However, the copula tail-value-at-risk with the symmetric marginal (normal and Student's t distribution) was not compared to an asymmetric distribution.

This work estimates the TVaR for bivariate stock data using copula. The C-TVaR estimates with symmetric marginal distributions like normal, Student's t, Cauchy, and logistic distributions for both bivariate stock data are compared with the asymmetric APTL distributions proposed by [14]. These are also compared with C-TVaR estimates with combinations of symmetric-asymmetric combinations marginals (like APTL-Student’s t and APTL-logistic). This is to determine the marginal distribution(s) that gives the minimum expected loss for investment of the bivariate stock portfolio. This innovative approach not only contributes to advancing the understanding of risk estimation in financial markets but also offers practical insights for portfolio managers and investors seeking to optimize risk-return profiles. By incorporating copula theory and exploring a diverse range of marginal distributions, this study aims to enhance the accuracy and reliability of TVaR estimation, thereby enabling more informed decision-making in portfolio management.

The remainder of the paper is organized as follows. Section 2 explains the TVaR estimation, copula functions and families, and the copula-TVaR model. Section 3 shows data application of the C-TVaR model on some sock portfolios. Finally, Section 4 concludes the paper.

2. METHOD

2.1. Copula Functions

Copulas are functions that enable the separation of the marginal distributions from the dependency structure of a given multivariate distribution [15]. Put differently; a copula helps isolate the joint or marginal probabilities of a pair of variables entangled in a more complex multivariate system. Sklar [16] introduced the application of copulas in multivariate modelling when he demonstrated that copulas can link the decomposition of multivariate distributions into marginal distributions. A p-dimensional copula $C(u_1, \ldots, u_p)$ is a multivariate distribution with uniform $U(0,1)$ marginal distribution. So every joint distribution $F(x_1, \ldots, x_p)$ whose marginal are given $F_1(x_1), \ldots, F_p(x_p)$ can be written as

$$F(x_1, \ldots, x_p) = C\{F_1(x_1), \ldots, F_p(x_p)\}. \quad (2)$$

2.1.1. Bivariate Copula Modeling

According to [17], the basic definition of a bivariate copula is a bivariate probability distribution function on [0, 1], for which the two univariate marginal distribution functions are uniform on [0, 1]. If the marginal distribution function of the continuous random vectors (X, Y) are F_X and F_Y, the copula bivariate probability distribution function can be expressed as

$$P(X \leq x, Y \leq y) = C(F_X(x), F_Y(y); \theta) = C(u_1, u_2), \quad (3)$$

for marginal probabilities

$$u_1 = F_X(x), u_2 = F_Y(y), \quad (4)$$

where C is a copula, θ is the copula parameter that measures the degree of association between two univariate CDFs (i.e., summarizes their dependence structure) , and x and y are the realizations of X and Y.

Families of copulas regularly used in multivariate dependence include the Elliptical and Archimedean copulas. The Elliptical copulas are derived from multivariate elliptical distributions, and the most critical copulas of this family are the Gaussian copula and the Student's t copula. On the other hand, some of the widely applied Archimedean copulas include the Clayton, Gumbel, and Frank copulas.

2.1.1.1. Elliptical Copula

The Gaussian and Student's t copulas are the most commonly applied bivariate copula in finance ([18], [19], [20], [21] and [22]). According to [22], the bivariate copula distribution function of the Gaussian family is expressed as follows

$$C(u,v) = \int_{-\infty}^{\Phi^{-1}(u)} \int_{-\infty}^{\Phi^{-1}(v)} \frac{1}{2\pi(1-\rho^2)} exp\left[-\frac{s^2-2\rho st+{u_2}^2}{v(1-\rho^2)}\right] dsdt, \quad (5)$$

where ρ is the correlation coefficient between the variables and, $\Phi^{-1}(u)$ and $\Phi^{-1}(v)$ represent the inverse of CDF of the standard normal variates, and ${T_v}^{-1}$ denotes the quantile function associated with the univariate distribution.

On the other hand, the Student's t copula incorporates the tail dependence at both the lower and upper tails. The degree of freedom parameter, v, affects the strength of the tail dependence. As v approaches infinity, the Student's t multivariate data converges to the Gaussian copula distribution. The function for the Student's t copula, as presented in Ferreira et al., 2016), is given by

$$C_{\rho,v}(u,v) = \int_{-\infty}^{{t_v}^{-1}} \int_{-\infty}^{{t_v}^{-1}} \frac{1}{2\pi(1-\rho^2)^{1/2}} exp\left[1+\left(\frac{x^2-2\rho xy+y^2}{v(1-\rho^2)}\right)^{-\left(\frac{v+2}{2}\right)}\right] dxdy. \quad (6)$$

The Student's t copula is often regarded as dominant in modeling non-linear and non-normal dependencies.

2.1.1.2. Archimedean Copula

Archimedean copulas are used to model asymmetric dependence structures. Some of the major copulas in the Archimedean family used in modeling financial data include the Clayton, Frank, and Gumbel copulas. Their copula function and joint density function, as presented in [23], are given below.

Clayton copula

The Clayton copula is asymmetric and measures dependence on the negative (left) tail. The Clayton copula function is given by

$$C_\lambda(u,v) = \left(u^{-\lambda}+v^{-\lambda}-1\right)^{-\left(\frac{1}{\lambda}\right)} \quad , \lambda \in [-1,\infty]\ and\ \alpha \neq 0, \quad (7)$$

where $u = F_{x_1}(x_1), v = F_{x_2}(x_2), \lambda$ is the parameter of the generating function $\varphi(t) = \frac{(t)^{-\lambda}-1}{\lambda}, \lambda \in [-1, \infty]$ that controls the dependence, $\lambda \to \infty$ entails perfect dependence, while $\lambda \to 0$ implies independence between the two random variables. So, the Clayton copula has more weight on the left tail. The generator uniquely defines the Gumbel copula, and *t* is a uniformly distributed random variable that varies from 0 to 1 regardless of whether it is equal to u or v

Gumbel copula

This copula is asymmetric and is sensitive to the right tail. So, the Gumbel copula measures dependency on the positive (right) tail. The function defines the Gumbel copula

$$C_\beta(u, v) = \exp\langle -\left[(-\ln(u))^\beta + (-\ln(v))^\beta\right]\rangle^{\frac{1}{\beta}} \quad , \beta \in [1, \infty], \tag{8}$$

where $u = F_{x_1}(x_1), v = F_{x_2}(x_2), \beta$ is the parameter of the generating function

$$\varphi(t) = (-\ln(t))^\beta, 0 < t < 1. \tag{9}$$

Frank copula

The function defines the Frank copula,

$$C_\theta(u, v) = -\frac{1}{\theta} ln\left(1 + \frac{(e^{-\theta u}-1)(e^{-\theta v}-1)}{e^{-\theta}-1}\right) \quad , \theta \in [-\infty, \infty]. \tag{10}$$

where $u = F_{x_1}(x_1), v = F_{x_2}(x_2), \theta$ is the parameter of the generating function

$$\varphi(x) = -ln\left(\frac{e^{-(\theta t)}-1}{e^{-(\theta)}-1}\right), \ 0 < t < 1. \tag{11}$$

2.2. Bivariate Copula-TVaR Model

This section presents the C-TVaR model for estimating the maximum loss expected for equally weighted bi-index stock portfolio investment at certain confidence levels over a specified period. Computing TVaR estimates is important because researchers have advised that computing VaR should not be enough as the losses beyond the given confidence level are not captured in VaR estimates. Given random losses X and Y of a bi-index portfolio, the bivariate C-TVaR model at a confidence level α for an equally weighted bi-index stock portfolio investment problem can be formulated as follows:

$$Minimize \ TVaR_{\gamma,\delta}(X, Y), \tag{12}$$

$$Subject \ to \ w_X = w_Y, \tag{13}$$

where

$$TVaR_{\gamma,\delta}(X, Y) = C\left(\frac{1}{1-\alpha}\int_{-\infty}^{-VaR_\gamma} xf(x)dx, \frac{1}{1-\alpha}\int_{-\infty}^{-VaR_\delta} yf(y)dy; \theta\right), \tag{14}$$

is the copula tail-value-at-risk of the portfolio at α level of significance from the univariate TVaR model in [24]. w_X and w_Y are the weights of risk of investment for stock indices X and Y, respectively, in the portfolio. $VaR_\gamma(x,\gamma)$ and $VaR_\gamma(x,\gamma)$ are the values at risk for each stock asset in the bi-index portfolio, with VaR given as $VaR_\gamma(x,\gamma) = F_{x,\gamma}^{-1}(1-\alpha)$ and $VaR_\delta(y,\delta) = F_{y,\delta}^{-1}(1-\alpha)$. $F_X^{-1}(1-\alpha)$ and $F_Y^{-1}(1-\alpha)$ are the inverse of the distribution function of the random losses X and Y for stock indexes 1 and 2 in the portfolio. γ and δ are the distribution parameters for index 1 and 2 return distributions in the portfolio. C is the copula function. θ is the copula parameter that measures the degree of association between two univariate CDFs (i.e summarizes their dependence structure).

To compute the copula-TVaR estimates using the bivariate copula model, the Gaussian, Student's t, Gumbel, Clayton, and Frank copulas are used to determine the best-fitted copula among them. The copulas are fitted on a bi-index stock portfolio, and the most suitable copula among them is determined using the AIC and BIC model selection criteria. The best-fitted copula model was then used to model the TVaR estimates of the portfolios with APTL marginal distribution and its combinations. This is then compared to TVaR models with normal, Student's t, Cauchy, and logistic distributions to determine if the APTL marginal estimates a smaller expected loss than the models with normal, Student's t, Cauchy, and logistic distributions. In copula-TVaR estimation, the marginal distribution or combination of distributions that estimates the smaller expected maximum loss at given confidence levels is considered the better distribution portfolio risk measurement (see [25]). Distributions that provide less risk of loss in an investment with unknown risk are preferable for risk-averse investors and risk manager use.

The distribution function of the APTL distribution proposed by [14] with parameters α, c and k, for $x \in \mathbb{R}$, is given by

$$F(x) = \begin{cases} \dfrac{\alpha^{1/1+\exp\left(-\frac{x-c}{k}\right)}-1}{\alpha-1}, & if\ \alpha > 0, \alpha \neq 1, c > 0, k > 0, \\ \dfrac{1}{1+\exp\left(-\frac{x-c}{k}\right)}, & if\ \alpha = 1. \end{cases} \tag{15}$$

On the other hand, for a random variable *x*, the distribution function, $F(x)$, of the normal, Student's t, Cauchy, and logistic distributions that have been used in modeling the univariate distribution of returns of the stock index is given as follows.

Normal distribution:

$$F(x) = \frac{1}{2}\left[1 + erf\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)\right], \tag{16}$$

where erf (z) = Gauss error function $\frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}\,dt$w.

Student's t distribution:

$$F(x) = \frac{1}{2} + x\Gamma\left(\frac{v+1}{2}\right) \times \frac{{}_2F_1\left(\frac{1}{2},\frac{v+1}{2};\frac{3}{2};\frac{x^2}{v}\right)}{\sqrt{\pi v}\,\Gamma\left(\frac{v}{2}\right)}, \tag{17}$$

where v is degrees of freedom.

Cauchy distribution:

$$F(x)=\frac{1}{\pi}arctan\left(\frac{x-\mu}{\beta}\right)+\frac{1}{2}, \tag{18}$$

where α is the location parameter, and β is the scale parameter and

$$F(x)=\frac{1}{1+exp\left(-\frac{x-\mu}{s}\right)}, \tag{19}$$

where location parameter $\mu \in \mathbb{R}$ and scale parameter $s > 0$.

3. APPLICATIONS, EMPIRICAL RESULTS, AND DISCUSSION

In this section, we investigate the flexibility of the APTL marginal distribution in bivariate C-TVaR estimation of financial portfolio data. This C-TVaR estimate for APTL marginal is compared to those with normal, Student's t, Cauchy, and logistic marginals. Some of the biggest stock indices in the world's top two biggest economies, the U.S. and China, are used. These include the Dow Jones Industrial Average (DJIA) index and Standard & Poor's 500 (S&P 500) Index in the United States of America and then the Shanghai Stock Exchange (SSE) and the Shenzhen Stock Exchange (SZSE) in China. The weekly price index of the DJIA and S&P500, SSE, and SZSE stock indices is collected from January 2007 to December 2022 from https://finance.yahoo.com.

The return series of each of the stock indices is obtained from their weekly price index and is log-transformed into their weekly log returns, R_t, using the formula

$$R_t = \ln\left(\frac{P_{t-1}}{P_t}\right), \tag{20}$$

where P_t is the opening price at period t and P_{t-1} is the opening price of the previous week. The log-returns of the stock indices are used to obtain the parameters of each stock index's parameters and derive the copula-TVaR estimates of the DJIA-S&P500 and SSE-SZSE portfolios using the TVaR model in section 2.3.

Table 1 presents the descriptive statistics of the DJIA, S&P500, SSE, and SZSE stock Indices**.** The skewness values of each stock index show that they are all negatively skewed. This indicates that all the indices had many small gains and fewer extreme losses (or negative returns) over the period under consideration.

Table 1. Univariate statistics of the DJIA, S&P500, SSE, and SZSE stock indices

	STATISTICS			
	SSE	**SZSE**	**DJIA**	**S&P**
Mean	0.03975	0.000585	0.001196	0.001176
Median	0.06026	0.001815	0.002722	0.002923
Std. Deviation	0.01455	0.03994	0.02542	0.025070
Minimum	-0.9243	-0.17150	-0.18440	-0.198700
Maximum	0.6225	0.12790	0.11070	0.130400
Skewness	-0.4779	-0.30140	-0.75570	-0.953200
Kurtosis	1.9478	1.51220	6.10690	9.786000

The SSE and SZSE stock Indices' kurtosis is less than 3. This indicates that the distribution of both SSE and SZSE stock indices is platykurtic. So, they have a flatter peak compared to the normal distribution. Meanwhile, the DJIA and S&P500 indices have kurtosis above 3. This indicates that the distribution of DJIA and S&P500 stock indices is leptokurtic. They are implying that there have been many price fluctuations in the past (positive or negative) away from the average price of the stocks. An investor might experience extreme price fluctuations for each of these stock indices.

3.1. Maximum Likelihood Estimates and Goodness of Fit

The Maximum Likelihood estimates and model selection of APTL distribution compared to the normal, student t, Cauchy, and logistic distributions are obtained for the DJIA, S&P500, SSE, and SZSE stock indices. The R statistical software is used, and the results are presented in Tables 2 and 3. The AIC and BIC model selection tests are carried out to determine which distribution best fits the returns of the different stock indices and are more suitable for the prediction of the return values. The AIC and BIC values for the DJIA, S&P500, SSE, and SZSE stock indices are shown in Table 2 and Table 3.

Table 2. Estimated model parameters and model selection criteria for DJIA and S&P500

Distribution	DJIA stock index			Model selection	
	Estimates			AIC	BIC
APTL (α, c, k)	0.01268	0.02746	0.01476	-3909.9	-3895.7
Normal (μ, σ)	0.001196	0.02540		-3755.6	-3746.1
Student t (a, c /df = v)	0.002430	0.01858	$v = 5.6107$	-3907.1	-3892.9
Cauchy (α , β)	0.003607	0.01130		-3803.9	-3794.5
Logistic (b, c)	0.002262	0.01281		-3886.4	-3876.9

Distribution	S&P stock index			Model selection	
	Estimates			AIC	BIC
APTL (α, c, k)	0.015430	0.02552	0.01420	-3961.2	-3947.0
Normal (μ, σ)	0.011760	0.02505		-3778.9	-3769.5
Student t (a, c / df = v)	0.002136	0.01788	$v = 5.5943$	-3969.0	-3954.9
Cauchy (α , β)	0.002975	0.01105		-3859.3	-3849.9
Logistic (b, c)	0.002005	0.01239		-3941.1	-3931.6

From Table 2, the APTL distribution has the smallest AIC and BIC values, but it is the second most appropriate for modeling the S&P-500 index, as the Student's t distribution has the smallest AIC and BIC values. This means the APTL distribution is the most suitable for modelling the DJIA stock index and predicting its return values. In contrast, the Student's t distribution is more appropriate for the S&P-500 index.

Table 3 shows the distribution best fitting the SSE and SZSE data using AIC and BIC model selection criteria. The result indicates that the APTL distribution has a significantly smaller AIC for the SSE index data than the normal, student t, Cauchy, and logistic distributions, but with the Logistic distribution having a smaller BIC value than APT, though not significant (since the difference is less than two units). Based on the AIC, it implies that the APTL distribution is significantly more appropriate for predicting the stock return values of SSE than the other distributions. On the other hand, based on the BIC, the logistic distribution is considered better for capturing the goodness of fit for the SSE index than APTL. Similarly, in the model selection of SZSE data, APTL has a significantly smaller AIC value than the other distributions but with a significantly bigger BIC than the Logistic distribution. This also implies that APTL will be significantly more appropriate in predicting the stock return values of SZSE while the logistic is a better goodness of fit distribution for SZSE. Hence, due to the discrepancies between the AIC and BIC, which is not usually the case, there is a need to investigate further if the APTL distribution will be better in predicting the maximum expected loss for equally weighted investment in the DJIA-S&P500 and SSE-SZSE portfolios at certain confidence levels. This is investigated by computing Copula-TVaR estimates for the DJIA-S&P500 and SSE-SZSE portfolios for different marginal. The marginals considered for the bivariate C-TVaR risk measurement model include the APTL-APTL, APTL-student's t, APTL-logistic, normal-normal, Student's t-student's t, Cauchy-Cauchy and logistic-logistic marginal distributions. Hence, it is necessary to estimate C-TVaR if the AIC and BIC of a given distribution, as seen in Table 3, are not both lower than those of other comparative distributions.

Table 3. Estimated model parameters and model selection criteria for SSE and SZSE

Distribution	SSE stock index			Model selection	
	Estimates			AIC	BIC
APTL (α, c, k)	0.046470	0.01190	0.008580	-4649.1	-4635.0
Normal (μ, σ)	0.039730	0.19680		-4590.7	-4581.3
Student t (a, c/df = v)	0.000470	0.01137	$v = 5.0604$	-4648.2	-4634.0
Cauchy (α , β)	0.0007819	0.007532		-4475.6	-4466.2
Logistic (b, c)	0.0004095	0.007779		-4645.3	-4635.8

Distribution	SZSE stock index			Model selection	
	Estimates			AIC	BIC
APTL (α, c, k)	0.0528800	0.03198	0.02366	-2985.7	-2971.6
Normal (μ, σ)	0.0005855	0.03992		-2940.5	-2931.0
Student t (a, c/df = v)	0.0014880	0.03186	$v = 5.2542$	-2982.4	-2968.3
Cauchy (α , β)	0.0028330	0.02089		-2799.8	-2790.3
Logistic (b, c)	0.0013370	0.02161		-2983.0	-2973.6

3.2. C-TVaR Estimation for Bi-Index Stock Portfolio Investment

In this section, we estimate the C-TVaR values of equally weighted risk bi-index portfolio investments using the model presented in section 2.3. This implies we are estimating the expected investment loss at the given confidence levels of the portfolio. We will use the DJIA-S&P500 and SSE-SZSE portfolio bi-index portfolios for this illustration. The Gaussian, Student's t, Gumbel, Clayton, and Frank copulas are used in modeling the DJIA-S&P500 and SSE-SZSE portfolios to determine the most suitable copula. The most suitable copula is then used to estimate the C-TVaR values of the portfolios with APTL marginal distributions. This also compares C-TVaR values with normal, Student's t, Cauchy, and logistic marginal distributions. The parameter estimates of the different copula models for the DJIA-S&P500 and SSE-SZSE portfolios, together with their AIC and BIC model selection values obtained using R statistical software, are presented in Table 4.

The results in Table 4 indicate that the estimate of the parameter of the Gaussian copula is positive. Consequently, the DJIA stock index is positively correlated with the S&P500 stock index. The table by the Gaussian copula also shows that the SSE and SZSE stock indices are strongly and positively correlated. Similar conclusions can be drawn based on the Student's t copula, as the associated parameter estimate is 0.9293. Again, the Student's t copula corresponds to the minimum AIC and BIC values among all the copulas considered in Table, implying that it is the best copula for modeling both the DJIA-S&P500 and SSE-SZSE stock portfolios among all the copulas under consideration. This makes it more suitable to model the dependence between the extreme value of asset returns for the DJIA-S&P500 and SSE-SZSE stock portfolios.

From the tail dependence for the different copulas presented in Table 5, the Gaussian and Frank copulas give zero as lower and upper tail dependence values. This is because neither copula captures tail dependence. The student-t copula captures the portfolio returns' upper and lower tail properties, with lower and upper tail values of 0.7560 and 0.6956 for the DJIA-S&P500 and SSE-SZSE portfolios, respectively. This can be one of the reasons the Student's t copula best fits the portfolio return data, as none of the other copulas captures both upper and lower tail dependence for the portfolio return. The estimate of the upper tail parameter for the Gumbel copula is 0.8050. As expected, the Gumbel copula assigns a higher probability to joint extreme positive events (upper tail dependence) on portfolio data.

Table 4. Parameters sstimates, AIC, and BIC of the copulas for the portfolios

	DJIA – S&P500 portfolio			
Copulas	**Parameter estimates**	**Log-likelihood**	**AIC**	**BIC**
Gaussian	0.9234	777.2	-1552.5	-1547.7
Student-t	0.9293/df=3.7226	821.6	-1639.1	-1629.7
Gumbel	3.89	766.5	-1531.0	-1526.3
Clayton	4.074	659.2	-1316.4	-1311.7
Frank	14.96	744.5	-1486.9	-1482.2
	SSE – SZSE portfolio			
Copulas	**Parameter estimates**	**Log-likelihood**	**AIC**	**BIC**
Gaussian	0.9234	777.2	-1552.5	-1547.7
Student-t	0.9293/df=3.7226	821.6	-1639.1	-1629.7
Gumbel	3.89	766.5	-1531.0	-1526.3
Clayton	4.074	659.2	-1316.4	-1311.7
Frank	14.96	744.5	-1486.9	-1482.2

Table 5. Tail dependence for the copulas

Copulas	**DJIA – S&P500 portfolio**		**SSE – SZSE portfolio**	
	Lower Tail dependence	**upper Tail dependence**	**Lower Tail dependence**	**upper Tail dependence**
Gaussian	0.0000	0.0000	0.000	0.000
Student-t	0.7560	0.7560	0.6956	0.6956
Gumbel	0.0000	0.8526	0.000	0.8050
Clayton	0.8948	0.0000	0.8435	0.000
Frank	0.0000	0.0000	0.000	0.000

On the other hand, we can see the Clayton copula having a lower tail value of 0.8435. So, the Clayton copula, when used in modeling the dependence of the SSE-SZSE stock portfolio, assigns a higher probability to joint extreme negative events (lower tail dependence). However, even though both estimates of Clayton and Gumbel copula are positive, the value of the Clayton copula is slightly higher than that of the Gumbel copula. This suggests a higher dependency during extreme losses and market crises than during a market boom. So, the SSE-SZSE stock index reacts similarly during crises and bear markets than during bull markets (market gains).

The copula tail value-at-risk estimates of the DJIA-S&P500 and SSE-SZSE portfolios are obtained using the most suitable copula for the bi-index stock portfolio, which is the Student's t copula. The marginal probability distributions used in C-TVaR estimation of the DJIA-S&P500 and SSE-SZSE bi-index portfolios include APTL/APTL, APTL/student-t, APTL/logistic, normal/normal, Student's t/student's t, Cauchy/Cauchy and logistic/logistic bivariate distributions. The C-TVaR estimates for these bivariate marginal are estimated at 99%, 99.5%, and 99.9% confidence levels, as also used in the work by Byun and Song(2021) and Bouye (2000). Emphasis will be placed on estimated values at a 99.9 % confidence level in making conclusions for each portfolio since it is the highest confidence level. This is because financial analysts recommend estimating portfolio risk at a very high level of confidence to ensure minimal significance, which provides optimal portfolio investment. The TVaR estimates for the DJIA-S&P500 and SSE-SZSE portfolios, for equal weights of investments, are obtained using the R statistics software as presented in Table 6.

The results of the copula tail-value-at-risk estimation for the DJIA-S&P500 portfolio show that the Student's t copula model with APTL-Student's t bivariate marginals gives the smallest tail value at risk for equally weighted investment in the DJIA-S&P500 portfolio. The estimates are 0.0948, 0.1058, and 0.1313 at the 99%, 99.5%, and 99.9% confidence levels, respectively. Taking the forecast at a 99.9% confidence level, it can concluded that for an equally weighted risky investment in the DJIA-S&P500 portfolio using the Student's t bivariate copula TVaR model with APTL-Student's t marginals, there is a 99.9% chance that the maximum expected weekly loss will not be more than 0.1313% of the invested amount due to the movement of market price. This means that 99.9% of the time, the maximum expected loss will be lower than 0.1313% of the investment value.

Meanwhile, for the SSE-SZSE portfolio, the Student's t bivariate copula TVaR model with APTL-APTL marginal gives the smallest expected losses of investment as 0.1235, 0.1391, and 0.1773 across the 99%, 99.5%, and 99.9% confidence levels respectively. So, based on the 99.9% confidence level, it can be concluded that using Student's t bivariate copula TVaR model with APTL-APTL marginals for an equally weighted risky investment in the SSE-SZSE portfolio, there is a 99.9% chance that the maximum expected weekly loss will not be more than 0.1773% of investment due to the movement of the market price. This means that 99.9% of the time, the maximum expected loss will be lower than 0.1773% of the investment value. The bivariate TVaR estimates of the DJIA-S&P500 and SSE-SZSE portfolios obtained in Table 6 will be challenging to get without the use of copula as the bivariate APTL, bivariate APTL-Student's t and APTL-Logistic distributions will be difficult to derive.

Table 6. C-TVaR estimates of S&P-DJIA and SSE-SZSE portfolios

DJIA-S&P500				
Copula	**Marginal Distribution**	**Confidence levels**		
		99%	**99.5%**	**99.9%**
Student-t	**APTL-APTL**	0.0951	0.1066	0.1352
	APTL-student's t	0.0948**	0.1058**	0.1313**
	APTL-logistic	0.1441	0.1628	0.2075
	Normal -Normal	0.1317	0.1430	0.1676
	Student's t-Student's t	0.1473	0.1724	0.2437
	Cauchy - Cauchy	7.5660	14.1195	58.8804
	Logistic - Logistic	0.1364	0.1538	0.1954

SSE-SZSE				
Copulae	**Marginal Distribution**	**Confidence levels**		
		99%	**99.5%**	**99.9%**
Student-t	**APTL-APTL**	0.1235**	0.1391**	0.1773**
	APTL-Student's t	0.1243	0.1865	0.2327
	APTL-logistic	0.1243	0.1404	0.1801
	Normal -Normal	0.1442	0.1566	0.1812
	Student's t-Student's t	0.1860	0.2185	0.3059
	Cauchy - Cauchy	10.0703	18.8041	78.3888
	Logistic - Logistic	0.1626	0.1830	0.2280

The comparison of the maximum expected loss for equally weighted investments in the DJIA-S&P500 portfolio and the SSE-SZSE portfolio based on 99.9% confidence level is presented in Table 7. This table shows that, at a 99.9% confidence level, the maximum expected loss for investing in an equally weighted DJIA-S&P500 portfolio is smaller than for the SSE-SZSE portfolio. Thus, it is slightly less risky to invest in the DJIA-S&P500 portfolio than in the SSE-SZSE portfolio.

Table 7. Comparison of TVaR Estimates Between DJIA-S&P500 and SSE-SZSE Portfolios

	PORTFOLIO	
	DJIA-S&P500	SSE-SZSE
Copula-TVaR	0.1313	0.1773

4. CONCLUSION

In this study, TVaR estimates for equally weighted risky bi-index portfolios were obtained using the Student's t bivariate copula model with APTL- APTL marginal compared to other combinations of marginal distributions. The APTL distribution, which has been used to estimate the value-at-risk of individual stock indices and provided better goodness of fit as well as smaller risk estimate than the normal, Student's t, Cauchy, and logistic distributions, is at this moment applied on portfolio C-TVaR estimates. Two of the largest stock indices in the world's top two economies – the DJIA and S&P500 stock indices in the U.S. and the SSE and SZSE stock indices in China are used for the illustration. The model selection results showed that the APTL distribution is more suitable for predicting stock returns of the DJIA, SSE, and SZSE stock index data stock returns than the normal, Student's t, Cauchy, and logistic distributions based on the AIC values. The Gaussian, Student's t, Gumbel, Clayton and Frank copulas were fitted to the bi-index portfolios – DJIA-S&P500 and SSE-SZSE. The copula model selection results showed that the Student's t copula, with the smallest AIC and BIC, is more suitable in both stock portfolios than the other copulas. Estimation of the tail-value-at-risk of the portfolios using the Student's t copula with APTL marginals and different combinations of marginals for the bi-index portfolio is carried out. Results obtained using the R software show that the expected maximum loss for an equally weighted investment in the DJIA-S&P500 at the 99%, 99.5%, and 99.9% confidence levels is smaller for APTL-student's t marginal than for the model with normal Student's t, Cauchy and logistic distributions. Meanwhile, the Student's t copula model with APTL-APTL marginals gives the smaller expected maximum loss for equally weighted investment in the SSE-SZSE portfolio beyond the 99%, 99.5%, and 99.9% confidence levels when compared to normal, Student's t, Cauchy and logistic distributions. From the results obtained, we can conclude that the APTL distribution or its combination with Student's t distribution presents a smaller expected loss of investment at given confidence levels than the normal, Student's t, Cauchy, and logistic distributions for DJIA-S&P500 and SSE-SZSE stock portfolios.

REFERENCES

[1] M. Y. T. Irsan and M. L. Sirait, "Value-at-risk (VaR) & Tailed value-at-risk (TVaR): Companies listed in LQ45," *International Journal of Economics, Commerce and Management,* vol. 8, no. 12, pp. 560-568, (2020.

[2] Z. Shen, L. Yukun and W. Chengguo, "Nonparametric inference for VaR, CTE and expectile with high-order precision," *North American Actuarial Journal,* vol. 23, p. 364–85, 2019.

[3] C. Bolance, M. Guillen and A. Padilla, Risk estimation using copulas. U.B. Risk centre working papers series 2015-01, Universitat de Barcelona, 2015.

[4] J. E. Methni, G. Laurent and G. Stephane, "Non-parametric estimation of extreme risk measures from conditional heavy-tailed distributions," *Scandinavian Journal of Statistics,* vol. 41, p. 988–1012, 2014.

[5] A. Castaner, M. M. Claramunt and M. Marmol, Tail value at risk: An analysis with the normal-power approximation. In Statistical and Soft Computing Approaches in Insurance Problems, pp. 87-112, Nova Science Publishers. ISBN 978-1-62618-506-7, 2013, pp. 87-111.

[6] M. Barges, H. Cossette and E. Marceau, "TVaR-based Capital Allocation with Copulas," *Insurance: Mathematics and Economics,* vol. 45, no. 3, pp. 348-361, 2009.

[7] V. Brazauskas, L. J. Bruce, L. P. Madan and Z. Riˇcardas, "Estimating Conditional Tail Expectation with Actuarial Applications in View," *Journal of Statistical Planning and Inference,* vol. 138, no. 3, p. 590–604, 2008.

[8] T. Kaiser and B. Vytaras, "Interval Estimation of Actuarial Risk Measures," *North American Actuarial Journal,* vol. 10, p. 249–268, 2006.

[9] R. Wang and Y. Wei, "Characterizing Optimal Allocations in Quantile-based Risk Sharing," *Insurance: Mathematics and Economics,* vol. 93, p. 288–300, 2020.

[10] R. Bairakdar, C. Lu and M. Melina, "Range Value-at-risk: Multivariate and Extreme Values," *arXiv .12473,* 2020.

[11] C. Bernard, K. Rodrigue and V. Steven, "Range Value-at-risk Bounds for Unimodal Distributions Under Partial Information," *Insurance: Mathematics and Economics,* vol. 94, p. 9–24, 2020.

[12] K. Syuhada, O. Neswan and B. P. Josaphat, "Estimating Copula-based Extension of Tail Value-at-risk and its Application in Insurance Claim," *Risk,* vol. 10, no. 113, pp. 1-26, 2023.

[13] S. Dutta and B. Suparna, "Nonparametric estimation of 100(1 - p)% expected shortfall: p as sample size is increased," *Communications in Statistics-Simulation and Computation,* vol. 47, p. 338–352, 2018.

[14] A. C. Iwuji, E. W. Okereke, B. Oruh and J. C. Nwabueze, "The Alpha Power Transformed Distribution: Properties, Application and VaR Estimation," *Indonesian Journal of Pure and Applied Mathematics,* vol. 5, no. 1, p. 82–98, 2023.

[15] M. Haugh, An Introduction to Copulas in IEOR E4602: Quantitative Risk Management, Lecture Notes. New York: Columbia University, 2016.

[16] M. Sklar, "Functions de répartition à n dimensions et leurs marges," *1959,* vol. 8, p. 229–231, Publ. Inst. Stat. Univ. Paris.

[17] B. Beare, "Copulas and Temporal Dependence.," *Econometrica,* vol. 78, p. 395–410, 2010.

[18] K. Byun and S. Song, "Value at Risk of Portfolios using Copulas," *Communications for Statistical Applications and Methods,* vol. 28, no. 1, pp. 59-79, 2021.

[19] R. A. Razak and N. C. Ismail, "Portfolio Risks of Bivariate Financial Returns using Copula-Var Approach: A Case Study on Malaysia and U.S. Stock Markets," *Global journal of pure and applied mathematics,* vol. 12, no. 3, pp. 1947-1964, 2016.

[20] M. Sahamkhadam, Copula-based portfolio optimization., Unpublished PhD thesis, Linnaeus University dissertations, Vaxjo. ISBN: 978-91-89283-79-4, 2021.

[21] N. Trabelsi and A. K. Tiwari, "Market Risk Optimization among Developed and Emerging Markets with CVaR Measure and Copula Simulation," *Risks,* vol. 7, no. 3, p. 78, 2019. https://doi.org/10.3390/risks7030078.

[22] P. H. Ferreira, R. L. Fiaccone, J. S. Lordelo, S. O. L. Sena and V. R. Duran, "Bivariate Copula-based Linear Mixed Effects Model: An Application to Longitudinal Child Growth Data," *Tendendas em matematica aplicada e computacional,* vol. 20, no. 1, p. 37–59, 2019.

[23] N. Salleh, F. Yusof and Z. Yusop, "Bivariate Copula Functions for Flood Frequency Analysis.," in *AIP conference proceedings 1750, 060007-1 – 060007-13*, 2016.

[24] M. Kemp, "VaR and Tail VaR Mindset: Presentation to VaR Open Forum," in *The Actuarial profession*, 2009 (March).

[25] D. L. Oslon and D. Wu, "The Impact of Distribution on Value-at-Risk Measure," *Mathematical and Computer Modelling,* vol. 58, pp. 1670-1676, 2013.