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The Alpha Power Transformed Logistic Distribution: Properties, Application and VaR Estimation

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Abstract

In this paper, a new three-parameter distribution, which is a member of the Alpha Power Transformed Family of distributions, is introduced. The new distribution is a generalization of the logistic model called the alpha power transformed logistic (APTL) distribution. Some mathematical properties of the new distribution like moments, quantile function, median, skewness, kurtosis, Rényi entropy, and order statistics are discussed. The parameters of the distribution are estimated using the maximum likelihood estimation method and a simulation study is performed to investigate the effectiveness of the estimates. The usefulness and flexibility of the APTL distribution in modelling financial data are investigated using two portfolio stock indices, namely the NASDAQ and New York stock indices, both from the United States stock market. Based on the model selection criteria, we are able to establish empirically that the APTL distribution is the best for modelling the two data sets, among the various distributions compared in the study. For each of the data, the quantile value-at-risk estimates for the APTL distribution give the smaller expected portfolio loss at high confidence levels in comparison to those of the other distributions.

Keywords: Alpha power transformed family of distributions; logistic distribution; maximum likelihood estimation; portfolio investments; value-at-risk.

Abstrak

Pada artikel ini, diperkenalkan distribusi baru dengan tiga parameter yang merupakan anggota dari keluarga distribusi Alpha Power Transformed. *Distribusi baru ini merupakan generalisasi dari model logistik yang disebut distribusi* Alpha Power Transform Logistics (APTL). *Selain itu, dibahas pula beberapa sifat matematika dari distribusi tersebut yaitu momen, fungsi kuantil, median, kemiringan, kurtosis, entropi Rényi, dan statistik terurut. Parameter distribusi diestimasi menggunakan metode* maximum likelihood estimation *dan studi simulasi dilakukan untuk menyelidiki keefektifan estimasi. Kegunaan dan fleksibilitas distribusi* APTL *dalam pemodelan data keuangan diselidiki menggunakan dua indeks saham portofolio dari pasar saham Amerika Serikat yaitu indeks saham* NASDAQ *dan* New York. *Berdasarkan kriteria pemilihan model, secara empiris, dihasilkan bahwa APTL adalah distribusi terbaik untuk memodelkan dua set data di antara berbagai distribusi yang dibandingkan pada penelitian ini. Untuk setiap data, estimasi kuantil* value-at-risk *untuk distribusi APTL memberikan kerugian portofolio yang diharapkan lebih kecil dengan tingkat kepercayaan tinggi dibandingkan dengan distribusi lainnya.*

Kata Kunci: *distribusi dari keluarga* Alpha power transformed; *distribusi logistik;* maximum likelihood estimation; *investasi portofolio*; value-at-risk.

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1. INTRODUCTION

Portfolio investments play an important role in meeting the financial needs of investors through the stock market. Efficient portfolio modelling and analysis attract more investment returns to individual investors, corporate planners, and government policymakers. It also leads to economic growth, mobilization of domestic savings as well as being a source of foreign investment to the country [1]. Financial data, such as portfolio assets price and returns, are mostly characterized by uncertainty [2][3][4]. Consequently, financial analysts use probability distributions to model asset prices and returns as well as in the estimation of the asset's risk. Value at Risk (VaR) is a financial metric that assists financial institutions and investors estimate the risk of an investment. It measures the amount of potential loss that could happen to a portfolio investment over a specified period of time. Financial institutions use VaR to determine the level of cash reserves in order to lower potential portfolio losses.

Different probability distributions have been used in the literature to model portfolio stock returns as well as value at risk. The logistic distribution is used in modelling financial stock data. In particular, they are used in the estimation of Value at Risk (VaR) due to their fat-tailed characteristic [7][8].

The cumulative distribution function (cdf) and probability density function (pdf) of a continuous random variable (X) that follows the logistic distribution with location parameter $c \in \mathbb{R}$ and scale parameter $k > 0$ are defined in (1) and (2) respectively [23].

$$G(x;c,k) = \frac{1}{1+e^{-\frac{x-c}{k}}} \qquad -\infty < x < \infty, \tag{1}$$

$$g(x;c,k) = \frac{e^{-\frac{x-c}{k}}}{k\left[1+e^{-\frac{x-c}{k}}\right]^2} \qquad -\infty < x < \infty. \tag{2}$$

A widely used approach to the computation of VaR is based on the quantile function [18]. Given $F_{X;\theta}$ as the distribution function of a random loss X, provided the inverse of the distribution function exists, the quantile-based value at risk at α a level of significance, as presented in [18], is obtained as $VaR_\alpha(x;\delta) = F^{-1}_{x;\delta}(1-\alpha)$, where δ is the distribution parameter or parameter vector, depending on whether the underlying distribution contains one parameter or more parameters. In the case of the logistic distribution, $\delta = (c,k)$ the VaR according to [23] is given by

$$F^{-1}(q) = c + k\, ln\left(\frac{q}{1-q}\right) \;, q \in (0,1).$$

Despite the wide acceptability of logistic distribution in modelling stock data, it has a major limitation. It is well known that the distribution is symmetric and leptokurtic. However, it lacks the asymmetry property possessed by several stock data. In recent years, there have been generalizations of the logistic distribution by many researchers. These include the Skewed logistic distribution and the type I generalized logistic distribution proposed in [9], Alpha –Skewed generalized Logistic distribution of type III in [10], exponentiated-exponential Logistic distribution [11], Gamma-logistic distribution [5], transmuted type II generalized logistic distribution [6], among others. Certainly, these generalizations have helped in achieving asymmetry, enhancing the flexibility of the concerned model, exploring tail properties, and improving the goodness of fit in several situations.

Amidst the different methods that have been used in generalizing univariate distributions like the logistic distribution, the Alpha power transformation method introduced by [12] has shown to be very efficient in incorporating skewness and increasing the flexibility of the baseline distributions.

For an arbitrary parent cumulative distribution function $G(x)$, the cdf of the Alpha Power Transformed – G family of distributions is given by

$$F(x;\alpha) = \begin{cases} \dfrac{\alpha^{G(x)} - 1}{\alpha - 1} & , if\ \alpha > 0, \alpha \neq 1 \\ G(x) & , if\ \alpha = 1 \end{cases}, \tag{3}$$

and its probability density function (pdf) is

$$f(x;\alpha) = \begin{cases} \dfrac{In\alpha}{\alpha - 1} g(x)\alpha^{G(x)}, if\ \alpha > 0, \alpha \neq 1 \\ g(x), if\ \alpha = 1 \end{cases}. \tag{4}$$

In the literature, some univariate probability distributions have been generalized using this Alpha Power Transformation method. Some of these distributions include the Alpha power transformed (APT) Pareto distribution (see [13]), the APT Lindley distribution (see [14]), the APT Weibull-G family of distributions (see [15]), the APT inverse Lomax distribution (see [16]) and APT log-logistic distribution (see [17]). To the best of our knowledge, this alpha power transformation technique is yet to be used to generalize the logistic distribution. This article extends the logistic distribution through the alpha power transformation method. Fundamental properties of the new distribution are extensively emphasized while attention is given to its financial time series applications, especially the computation of the value at risk (VaR).

The rest of this paper is designed as follows. Section 2 presents the Alpha power transformed logistic (APTL) distribution. Moments and other properties of the distribution are derived in Section 3. Section 4 is dedicated to the maximum likelihood estimation of the parameters of the distribution. A simulation study based on the model is carried out in Section 5. In Section 6, the distribution is applied to real data sets. Value-at-risk estimates of the distribution are obtained in section 7. The conclusion of this article is presented in Section 8.

2. THE APTL DISTRIBUTION

The random variable (r.v) X is said to have an APTL distribution with three parameters α, c, and k if the cdf of X, for $x \in \mathbb{R}$, is obtained by substituting (1) into (3), as presented in (5)

$$F(x)=\begin{cases}\dfrac{\alpha^{\left(\frac{1}{1+e^{-\frac{x-c}{k}}}\right)}-1}{\alpha-1}, & \text{if } \alpha>0, \alpha\neq 1, c\in\mathbb{R}, k>0,\\[2ex] \dfrac{1}{1+e^{-\frac{x-c}{k}}}, & \text{if } \alpha=1.\end{cases} \tag{5}$$

The pdf that corresponds to (5) is given by

$$f(x)=\begin{cases}\dfrac{In\alpha}{\alpha-1}\left[\dfrac{e^{-\frac{x-c}{k}}}{k\left[1+e^{-\frac{x-c}{k}}\right]^2}\right]\alpha^{\left(\frac{1}{1+e^{-\frac{x-c}{k}}}\right)}-1, & \text{if } \alpha>0, \alpha\neq 1, c\in\mathbb{R}, k>0.\\[2ex] \dfrac{e^{-\frac{x-c}{k}}}{k\left[1+e^{-\frac{x-c}{k}}\right]^2}, & \text{if } \alpha=1.\end{cases} \tag{6}$$

Figure 1 contains the graphs of the pdf of APTL distributions for various values of its parameters. It follows that the distribution can be right-skewed and unimodal.

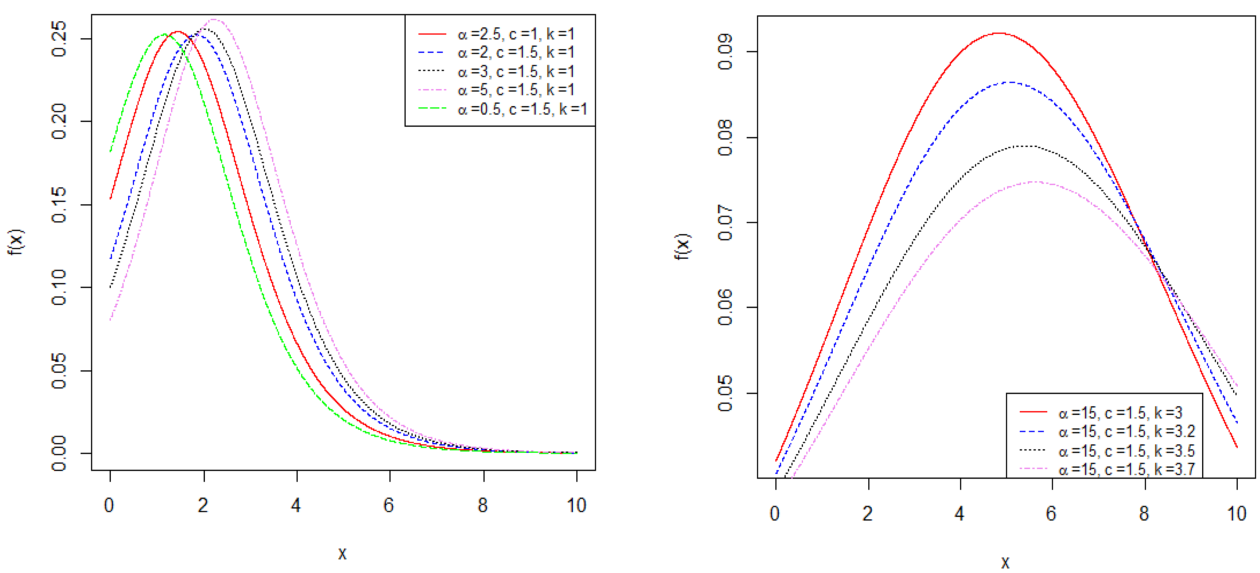

Figure 1. Plots of the pdf of the APTL distribution for various values of its parameters.

If X has an APTL distribution with the cdf and pdf stated in (5) and (6) respectively, we write $X \sim APTL(\alpha, c, k)$. For the APTL distribution, the survival function $S(x)$, and the hazard rate function $h(x)$ are given by

$$S(x) = 1 - F(x) = \frac{\alpha - \alpha^{\left(1+e^{-\left(\frac{x-c}{k}\right)}\right)^{-1}}}{\alpha - 1}, \tag{7}$$

and

$$h(x) = \frac{f(x;\alpha,c,k)}{S(x)} = \frac{In\alpha e^{-\left(\frac{x-c}{k}\right)} \alpha^{\left(1+e^{-\left(\frac{x-c}{k}\right)}\right)^{-1}}}{k\left[1+e^{-\left(\frac{x-c}{k}\right)}\right]^{2}\left[\alpha - \alpha^{\left(1+e^{-\left(\frac{x-c}{k}\right)}\right)^{-1}}\right]}. \tag{8}$$

In Figure 2, we have the plots of the hazard rate function (h(x)) corresponding to various values of the APTL distribution. So far, we can say that the hazard rate function for APTL distribution can be an increasing function of x.

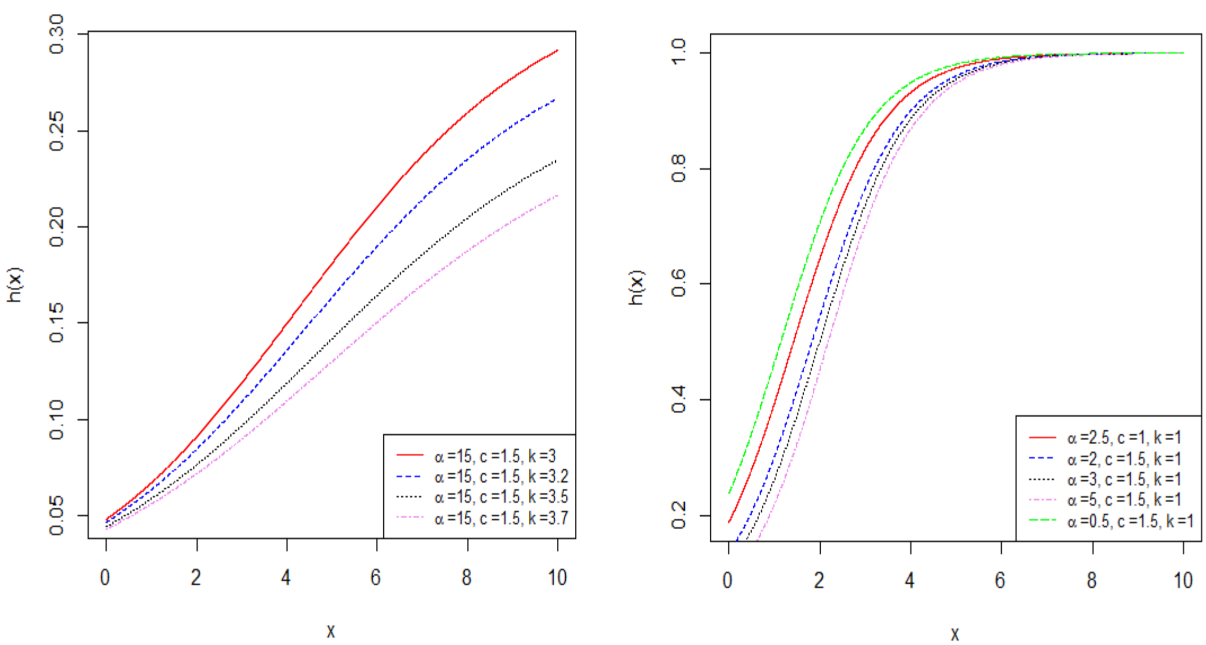

Figure 2. Plots of the hazard rate function of the APTL distribution for various values of its parameters

3. PROPERTIES OF THE APTL DISTRIBUTION

This section deals with some statistical properties of the APTL distribution.

3.1 Moments

Theorem 1. Let X be a random variable (r.v) from the APTL distribution, then the r*th* moment of X is

$$\mu_r^{'} = \omega_i \left[c^r B(2,b) + k^r \int_0^{\infty} y^r \frac{e^{-y}}{\left[1+e^{-y}\right]^{i+2}} dy \right], \tag{9}$$

where $\omega_i = \frac{ln(\alpha)}{(\alpha-1)} \sum_{i=0}^{\infty} \frac{(\log(\alpha))^i}{i!}$ and $y = \frac{x-c}{k}$.

Proof.

The rth moment of a random variable X is given as

$$\mu_r^{'} = \int_0^{\infty} x^r f(x;c,k,\alpha) dx .$$

The pdf of APTL distribution in (6) is given as

$$f(x;c,k,\alpha)dx = \frac{In\alpha}{\alpha-1} \left[\frac{e^{-\frac{x-c}{k}}}{k\left[1+e^{-\frac{x-c}{k}}\right]^2} \right] \alpha^{\left[\frac{1}{1+e^{-\frac{x-c}{k}}}\right]} \text{ for } \alpha > 0, \alpha \neq 1, c \in \mathbb{R}, k > 0 .$$

Using the series representation of the exponential function, $z^{\delta} = \sum_{i=0}^{\infty} \frac{(logz)^i}{i!} \delta^i$, the pdf can be expressed as follows:

$$f(x;c,k,\alpha) = \frac{ln(\alpha)}{(\alpha-1)} \sum_{i=0}^{\infty} \frac{\log(\alpha)^i}{i!} \frac{e^{-\left(\frac{x-c}{k}\right)}}{k\left[1+e^{-\left(\frac{x-c}{k}\right)}\right]^{i+2}} = \omega_i \frac{e^{-\left(\frac{x-c}{k}\right)}}{k\left[1+e^{-\left(\frac{x-c}{k}\right)}\right]^{i+2}} ,$$

where $\omega_i = \frac{ln(\alpha)}{(\alpha-1)} \sum_{i=0}^{\infty} \frac{(\log(\alpha))^i}{i!}$.

So $\mu_r' = \int_0^\infty x^r f(x;c,k,\alpha)dx = \omega_i \int_{-\infty}^\infty x^r \frac{e^{-\left(\frac{x-c}{k}\right)}}{k\left[1+e^{-\left(\frac{x-c}{k}\right)}\right]^{i+2}}$. Let $y = \frac{x-c}{k}$ $, x = yk + c$, $dx = kdy$.

$$\mu_r' = \omega_i \int_0^\infty (yk+c)^r \frac{e^{-y}}{k\left[1+e^{-y}\right]^{i+2}} kdy = \omega_i \left[c^r B(2,b) + k^r \int_0^\infty y^r \frac{e^{-y}}{\left[1+e^{-y}\right]^{i+2}} dy \right]$$

where $\omega_i = \frac{ln(\alpha)}{(\alpha-1)} \sum_{i=0}^\infty \frac{(\log(\alpha))^i}{i!}$. This completes the proof. ■

The mean of the APTL distribution is obtained by putting r = 1.

$$\mu_1' = \omega_i \left[c\mathrm{B}(2,b) + k \int_0^\infty y \frac{e^{-y}}{\left[1+e^{-y}\right]^{i+2}} dy \right]. \tag{10}$$

The variance of the APTL distribution is given as

$$\sigma^2 = \mu_2' - (\mu_1')^2 = \omega_i \left[c^2 B(2,b) + k^2 \int_0^\infty y^2 \frac{e^{-y}}{\left[1+e^{-y}\right]^{i+2}} dy \right] - (\mu_1')^2. \tag{11}$$

Figure 3 depicts the mean and variance of the APTL distribution based on the chosen values of the parameters.

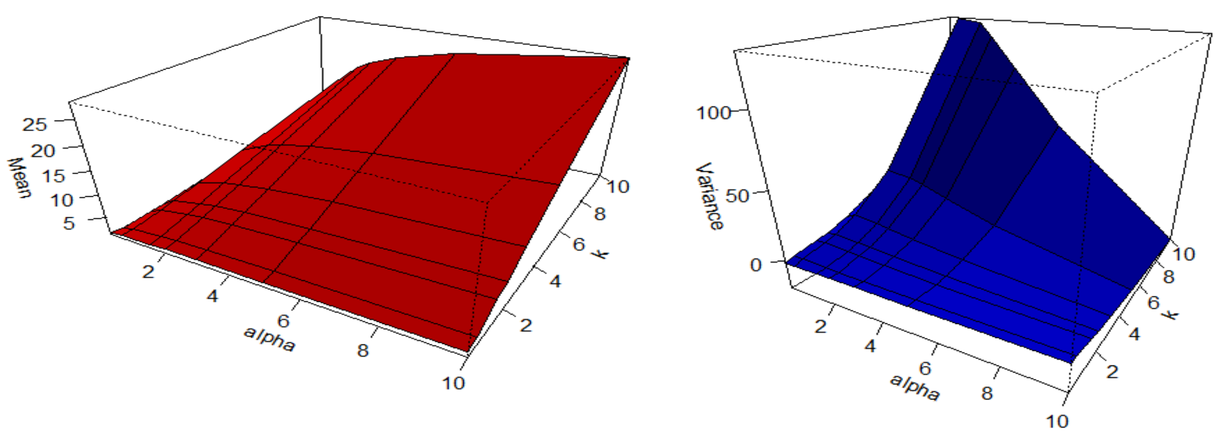

Figure 3. Plots of the mean and variance of the APTL distribution.

3.2 Quantile function and the related measures of skewness and kurtosis

The quantile function (x_q) of the APTL distribution is determined using (12).

$$\frac{\alpha^{\left[\frac{1}{1+e^{-\frac{x_q-c}{k}}}\right]}-1}{\alpha-1}=q\text{, where } q\epsilon(0,1). \tag{12}$$

Thus, the qth quantile function is given by

$$x_q=c-klog_e\left\{\frac{log_e(\alpha)-log_e\left[\left(1+q(\alpha-1)\right)\right]}{log_e\left[1+q(\alpha-1)\right]}\right\}. \tag{13}$$

The median can be obtained as

$$x_{0.5}=c-klog_e\left\{\frac{log_e(\alpha)-log_e\left[\left(1+0.5(\alpha-1)\right)\right]}{log_e\left[1+0.5(\alpha-1)\right]}\right\}. \tag{14}$$

The analysis of the shape of the APTL distribution can be performed by the study of skewness and kurtosis. The skewness (S_k) and kurtosis (K) of the APTL distribution are obtained using Bowley's coefficient of skewness and Moor's coefficient of Kurtosis respectively as presented in [16]. They are given by

$$S_k=\frac{x_{0.75}-2x_{0.5}+x_{0.25}}{x_{0.75}-x_{0.25}}, \tag{15}$$

and

$$K=\frac{x_{0.875}-x_{0.625}-x_{0.375}+x_{0.125}}{x_{0.75}+x_{0.25}}. \tag{16}$$

The plots of the skewness and kurtosis are presented in Figure 4. Consequently, the distribution can be left-skewed or right-skewed. For the plotted values, the graphical representation of the Moors' kurtosis indicates that the APTL distribution can also be platykurtic.

3.3 Rényi Entropy

For a given pdf, the Rényi entropy is defined by

$$I_R(\lambda)=\frac{1}{1-\lambda}log\int_0^\infty f(x)^\lambda\, dx \quad , \lambda>0,\lambda\neq 1. \tag{17}$$

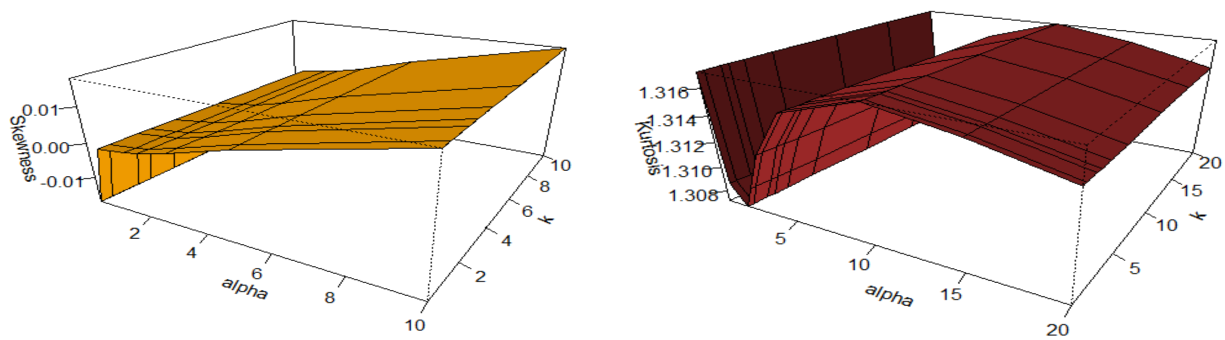

Figure 4. Bowley's skewness and Moors' kurtosis plots for APTL distribution for c = 1.

For the APTL distribution, the function $f(x)^{\lambda}$ can be written as

$$f(x)^{\lambda}=\left(\frac{In\alpha}{\alpha-1}\right)^{\lambda}\left[\frac{e^{-\left(\frac{x-c}{k}\right)}}{k\left[1+e^{-\left(\frac{x-c}{k}\right)}\right]^{2}}\right]^{\lambda}\alpha^{\left[\frac{1}{1+e^{-\left(\frac{x-c}{k}\right)}}\right]\lambda}.$$

Let $y=\dfrac{x-c}{k}$ $,dx=kdy$. Then $f(x)^{\lambda}=\left(\dfrac{In\alpha}{\alpha-1}\right)^{\lambda}\left[\dfrac{e^{-y}}{k\left[1+e^{-y}\right]^{2}}\right]^{\lambda}\alpha^{\left[1+e^{-y}\right]-\lambda}$.

Applying $z^{\delta}=\sum_{i=0}^{\infty}\dfrac{(logz)^{i}}{i!}\delta^{i}$ and $(1+x)^{-n}=\sum_{j=0}^{\infty}\binom{-n}{j}x^{j}$.

We get

$$f(x)^{\lambda}=\sum_{i,j=0}^{\infty}\left(\frac{kln\alpha}{k(\alpha-1)}\right)^{\lambda}\frac{(log\alpha)^{i}}{i!}\binom{-\lambda(i+2)}{j}e^{-y(j+\lambda)},$$

where $t_{i,j}=\left(\dfrac{kln\alpha}{k(\alpha-1)}\right)^{\lambda}\dfrac{(log\alpha)^{i}}{i!}\dbinom{-\lambda(i+2)}{j}$.

So $I_R(\lambda)=\dfrac{1}{1-\lambda}\log\sum_{i,j=0}^{\infty}\left(\dfrac{kln\alpha}{k(\alpha-1)}\right)^{\lambda}\dfrac{(log\alpha)^{i}}{i!}\dbinom{-\lambda(i+2)}{j}\int_{0}^{\infty}e^{-y(j+\lambda)}kdy$.

Applying the definite integral: $\int_0^\infty e^{-px}dx = \frac{1}{p}$, $p > 0$.

Then, the Rényi entropy is

$$I_R(\lambda) = \frac{k^{1-\lambda}}{1-\lambda} \log \sum_{i,j=0}^{\infty} \left(\frac{\ln(\alpha)}{(\alpha-1)} \right)^{\lambda} \frac{(log\alpha)^i}{i!(j+\lambda)} \binom{-\lambda(i+2)}{j} \qquad ,(\lambda+j) > 0. \tag{18}$$

3.4 Order Statistics

Let $X_1, X_2, X_3, \dots, X_n$ be a random sample from the APTL distribution with their respective order statistics $X_{(1)}, X_{(2)}, X_{(3)}, \dots, X_{(n)}$. Then the pdf of the kth order statistic $X_{(k)}$, is calculated as

$$f_{x_k}(x) = \frac{\ln(\alpha) n! (\Omega-1)}{(k-1)!(n-k)!(\alpha-1)^{n+2} k\Omega^2} \left(\alpha^{\Omega^{-1}} - 1 \right)^{k-1} \left(\alpha - \alpha^{\Omega^{-1}} \right)^{n-k}, \tag{19}$$

where $\Omega = 1 + e^{-\left(\frac{x-c}{k}\right)}$.

The pdfs of the first and nth-order statistics of the APTL distribution are respectively given as

$$f_{x_1}(x) = \frac{\ln(\alpha) n (\Omega-1)}{(n-1)!(\alpha-1)^{n+2} \Omega^2} \left(\alpha - \alpha^{\Omega^{-1}} \right)^{n-1}, \tag{20}$$

and

$$f_{x_n}(x) = \frac{\ln(\alpha) n (\Omega-1)}{(n-1)!(\alpha-1)^{n+2} \mathrm{n}\Omega^2} \left(\alpha^{\Omega^{-1}} - 1 \right)^{n-1}. \tag{21}$$

4. ESTIMATIONS

This section discusses the parameter estimation of the APTL distribution using the maximum likelihood (ML) method. The ML method is widely used due to its desirable properties namely consistency, asymptotic efficiency, and invariance (see [18]). In addition, the ML method has been proven to outperform other methods of estimating parameters (see [16]).

4.1 ML Estimation

Let $X_1, X_2, , \dots, X_n$ be observed values from the APTL distribution. Given pdf

$$f\left(x,c,k,\alpha\right)=\frac{In\alpha}{\alpha-1}\left[\frac{e^{-\frac{x-c}{k}}}{k\left[1+e^{-\frac{x-c}{k}}\right]^{2}}\right]\alpha^{\left[\frac{1}{1+e^{-\frac{x-c}{k}}}\right]}. \tag{21}$$

The likelihood function = $l\left(c,k,\alpha;x\right)=\prod_{i=1}^{n}f\left(x_i;c,k,\alpha\right)$

$$=\prod_{i=1}^{n}\left[\frac{In\alpha}{\alpha-1}\left[\frac{e^{-\frac{x-c}{k}}}{k\left[1+e^{-\frac{x-c}{k}}\right]^{2}}\right]\alpha^{\left[\frac{1}{1+e^{-\frac{x-c}{k}}}\right]}\right]. \tag{22}$$

The maximum likelihood estimators (MLEs) of the proposed model parameters c,k and α are obtained using the log-likelihood function given

$$lnL=\ell=nln\left[ln\left(\alpha\right)\right]-n\ln\left(\alpha-1\right)-\sum_{i=1}^{n}\left(\frac{x_i-c}{k}\right)-nlnk-2\sum_{i=1}^{n}\left(1+e^{-\left(\frac{x_i-c}{k}\right)}\right)+\sum_{i=1}^{n}\left[\frac{1}{1+e^{-\left(\frac{x_i-c}{k}\right)}}\right]ln\alpha . \tag{23}$$

The derivatives of (23) with respect to c,k and α are given by

$$\frac{d\ell}{dc}=\frac{n}{k}+2\sum_{i=1}^{n}\left[\frac{e^{-\left(\frac{x_i-c}{k}\right)}}{k\left[1+e^{-\left(\frac{x_i-c}{k}\right)}\right]}\right]+\ln\left(\alpha\right)\sum_{i=1}^{n}\left[\frac{e^{-\left(\frac{x_i-c}{k}\right)}}{k\left[1+e^{-\left(\frac{x_i-c}{k}\right)}\right]^{2}}\right], \tag{24}$$

$$\frac{d\ell}{dk}=\sum_{i=1}^{n}\left[\frac{\left(x_i-c\right)}{k^2}\right]-\frac{n}{k}+2\sum_{i=1}^{n}\left[\frac{\left(x_i-c\right)e^{-\left(\frac{x_i-c}{k}\right)}}{k^2\left[1+e^{-\left(\frac{x_i-c}{k}\right)}\right]}\right]+\ln\left(\alpha\right)\sum_{i=1}^{n}\left[\frac{\left(x_i-c\right)e^{-\left(\frac{x_i-c}{k}\right)}}{\left[k\left\{1+e^{-\left(\frac{x_i-c}{k}\right)}\right\}\right]^{2}}\right], \tag{25}$$

$$\frac{d\ell}{d\alpha}=\frac{n}{\alpha ln\left(\alpha\right)}-\frac{n}{\alpha-1}+\frac{1}{\alpha}\sum_{i=1}^{n}\left[\frac{1}{1+e^{-\left(\frac{x_i-c}{k}\right)}}\right]. \tag{26}$$

The ML estimators of the parameters c, k and α are obtained by equating (24), (25), and 26) to zero and solving simultaneously. At this point, it is obvious that only a numerical approach can be used to solve the equations. Interestingly, the R package is useful in obtaining the solution.

5. SIMULATION STUDY

In this section, the parameter estimates of the APTL distribution, the mean square error as well as the bias measure are computed using a simulation study. To achieve this, 1000 random samples from different sample sizes $n = 25, 50, 100, \text{and } 150$ are generated from the APTL distribution. Two sets of parameters as assigned as follows; Set 1. $\hat{\alpha} = 0.5, \hat{c} = 0.3,\ \hat{k} = 0.5$ and Set 2. $\hat{\alpha} = 2, \hat{c} = 3,\ \hat{k} = 4.5$. The average estimates, MSEs, and Bias for the different sample sizes for sets 1 and 2 using the R package are presented in Table 1.

Table 1. Estimates, MSEs, and bias of APTL distribution for 2 sets of parameters.

Set 1. $\hat{\boldsymbol{\alpha}} = \mathbf{0.5}, \hat{\boldsymbol{c}} = \mathbf{0.3},\ \hat{\boldsymbol{k}} = \mathbf{0.5}$

n	Average Estimate			MSEs			Bias		
	$\hat{\alpha}$	$\hat{c}$	$\hat{k}$	$\hat{\alpha}$	$\hat{c}$	$\hat{k}$	$\hat{\alpha}$	$\hat{c}$	$\hat{k}$
25	0.7949	0.2216	0.4420	0.2020	0.1231	0.0012	0.2949	-0.0784	-0.0580
50	0.5356	0.4020	0.5117	0.1840	0.0937	0.0012	0.0356	0.1020	0.0117
100	0.5703	0.4204	0.5084	0.1788	0.8692	0.0043	0.0703	0.1204	0.0084
150	0.5017	0.4401	0.5145	0.1651	0.8446	0.00	0.0017	0.1401	0.0145

Set 1. $\hat{\boldsymbol{\alpha}} = \mathbf{2}, \hat{\boldsymbol{c}} = \mathbf{3},\ \hat{\boldsymbol{k}} = \mathbf{4.5}$

n	Average Estimate			MSEs			Bias		
	$\hat{\alpha}$	$\hat{c}$	$\hat{k}$	$\hat{\alpha}$	$\hat{c}$	$\hat{k}$	$\hat{\alpha}$	$\hat{c}$	$\hat{k}$
25	0.7085	1.9432	4.1290	0.4274	3.3505	0.4125	0.5085	-0.0568	-0.3710
50	0.2200	5.1081	4.6335	0.1532	22.221	0.0891	0.0200	2.1081	0.1335
100	0.3818	3.7061	4.5121	0.1920	16.4531	0.0048	0.1818	0.7061	0.0121
150	0.3624	3.1486	4.5300	0.1532	0.9273	0.0329	0.1624	0.1486	0.0300

6. APPLICATION TO DATA

This section shows how effective and flexible the APTL model is in modelling financial data when compared to some existing distributions. The log-returns of a portfolio of stock index are used in determining this. Two sets of portfolio stock indices, the New York Stock Index and the NASDAQ stock index are used. These two stock indices are used in this analysis because they were listed as the two most valuable stock indices in the world, by market capitalization, as of December 2020 by some leading stock investment websites such as www.caproasia.com, www.markets.businessinsider.com, and www.ig.com. Data of weekly opening stock prices of the New York Stock Index and NASDAQ stock index from 01/01/2007 to 31/12/2020 is collected from https://finance.yahoo.com. Each of the two data sets is converted into weekly log returns using the formula

$$R_t = \log\left(\frac{P_{t-1}}{P_t}\right),$$

where P_t is the closing price at period t.

Table 2 contains descriptive statistics for the log-returns of the two data. It is crystal clear that the returns of both data are negatively skewed and leptokurtic.

Table 2. Descriptive Statistics for log returns of NASDAQ and New York Data.

Log Returns	Mean	Standard Deviation	Skewness	Kurtosis
NASDAQ	0.0010	0.0124	-0.7094	7.0239
New York Stock	0.0003	0.0120	-1.1696	13.1887

Furthermore, we fit each of the APTL distribution, normal distribution, logistic distribution, Kumaraswamy logistic, odd Lindley logistic, and Cauchy distributions to each of the transformed data. In each case, we compare the fits of the distributions using the Akaike Information Criterion (AIC), Consistent Akaike Information criterion (CAIC), Hannan-Quinn Information criterion (HQIC), and Bayesian Information criterion (BIC). Using notations, according to [22], are given by

$$AIC = -2\ell(\theta) + 2m, \tag{27}$$

$$BIC = -2\ell(\theta) + m \times ln(n), \tag{28}$$

$$CAIC = -2\ell(\theta) + m[ln(n) + 1], \tag{29}$$

$$HQIC = -2\ell(\theta) + 2m \times ln[ln(n)], \tag{30}$$

where $\ell(\theta)$ is the log-likelihood function, m is the number of parameters and n is the sample size.

The ML estimates for the competitive distributions and the numerical results of some measures of goodness of fit for the log of returns of NASDAQ stock index data obtained using the R statistical software are shown in Table 3. Table 4 also presents the ML estimates and the numerical results for some measures of goodness of fit for all the competitive models for New York stock index data.

Table 3. Estimated model parameters, LL, and goodness of fit measures for the NASDAQ stock index.

						The Goodness of Fit Measures			
Distribution	Estimates				ℓ	AIC	CAIC	BIC	HQIC
APTL (α, c, k)	0.0098	0.014	0.0074		2224.7	- 4443.4	-4426.6	- 4429.6	- 4438.0
Cauchy (α, β)	0.0022	0.0059			2160.9	- 4317.8	-4306.6	- 4308.6	- 4313.8
Logistic (b, c)	0.0016	0.0063			2213.9	- 4423.8	-4412.7	- 4414.7	- 4420.3
Kumaraswamy Logistic (a, b, c, d)	0.0061	0.0062	0.795	1.2930	2220.7	- 4433.4	-4411.0	- 4415.0	- 4426.3
Odd Lindley logistic (α, β, θ)	21.59	0.0473	0.0133		2095.4	- 4184.7	-4168.0	- 4171.0	- 4179.4
Normal (μ, σ)	0.0010	0.0124			2166.1	- 4328.2	-4324.6	- 4319.0	- 4324.6

Table 4. Estimated model parameters, LL, and goodness of fit measures for the New York stock index.

						The Goodness of Fit Measures			
Distribution	**Estimates**				ℓ	**AIC**	**CAIC**	**BIC**	**HQIC**
APTL (α, c, k)	0.0113	0.0123	0.0067		2290.1	- 4574.1	-4557.4	- 4560.4	- 4568.8
Cauchy (α, β)	0.0013	0.0048			2267.7	- 4531.3	-4520.1	- 4522.1	- 4527.8
Logistic (b, c)	0.0008	0.0058			2276.0	- 4548.0	-4536.8	- 4538.8	- 4544.5
Kumaraswamy Logistic (a, b, c, d)	0.0046	0.0056	0.8093	1.2742	2282.0	- 4556.1	-4533.7	- 4537.9	- 4549.0
Odd Lindley logistic (α, β, θ)	21.57	0.0477	0.0138		2095.9	- 4185.8	-4169.1	- 4172.1	- 4180.5
Normal (μ, σ)	0.0003	0.0120			2191.9	- 4379.9	-4368.7	- 4370.7	- 4376.3

The lower the values of the AIC, BIC, CAIC, and HQIC the better the fit to the data. Based on the results in Tables 3 and 4, we can see that the APTL model has the smallest values of AIC, CAIC, HQIC, and BIC. According to these criteria, we conclude that the APTPL model is the best-fitted model compared to the other competitive models. Plots of the estimated densities and cdfs for NASDAQ and New York data are respectively given in Figures 5 and 6. The figures all indicate that the APTL distribution fits both data well.

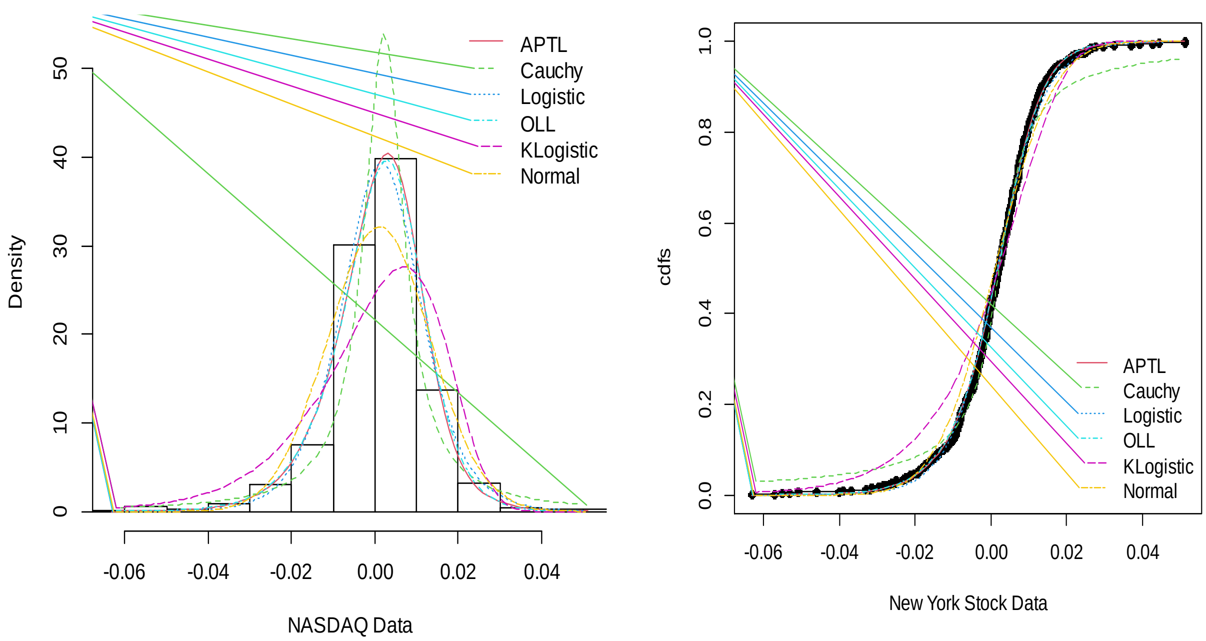

Figure 5. Plots of the estimated pdfs (left panel) and cdfs (right panel) for the log returns of NASDAQ data.

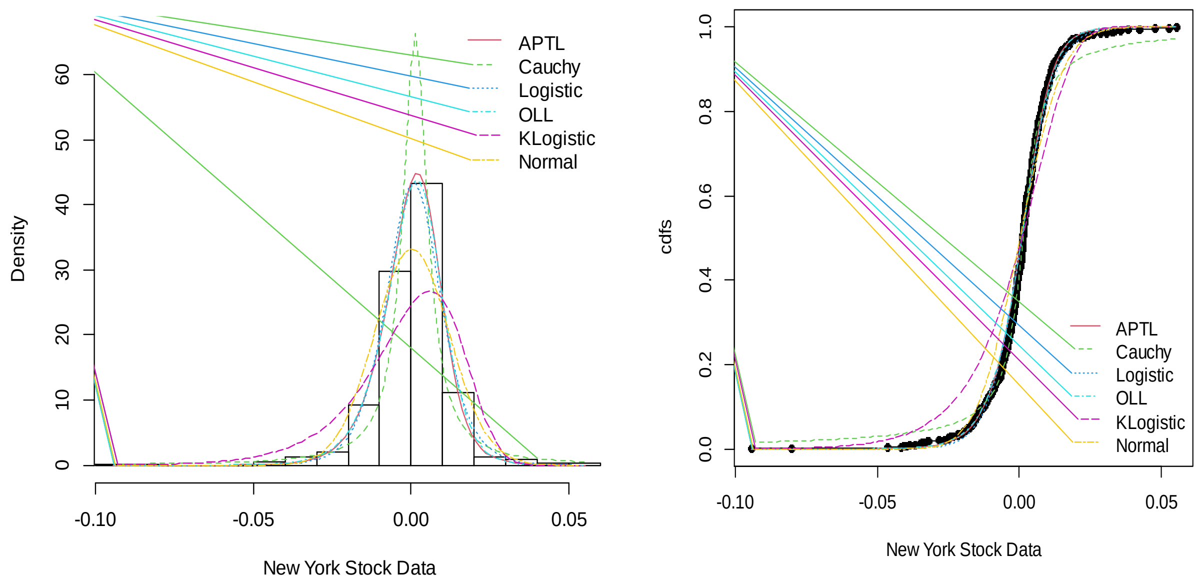

Figure 6. Plots of the estimated pdfs (left panel) and cdfs (right panel) for the log returns of New York stock data.

7. ESTIMATION OF QUANTILE-BASED VALUE AT RISK (VAR)

We estimate the quantile-VaR for the APTL distribution and the other competing distributions by replacing their parameters with their respective maximum likelihood estimates obtained from the NASDAQ and New York (NY) stock data. The 95%. 96%, 97%,98%, and 99% confidence levels quantile-based VaR estimates for the NASDAQ stock index data and New York stock index data are shown in Table 5. From Table 5, it is obvious that the higher the level of confidence (like 98% and 99%), the lower the VaR estimate corresponding to the APTL distribution than those of normal, t, and Cauchy distributions for both the NY and NASDAQ stock indices. This implies the APTL distribution gives lower expected portfolio loss at higher confidence levels than the normal, t, and Cauchy distributions. Since financial institutions and regulators use VaR estimates based on a 99% confidence level [21], it is then evident that the APTL distribution is preferable for use by financial regulators in estimating value at risk compared to the logistic, Normal, and Cauchy distributions. So from the VaR estimates obtained, based on APTL distribution, we can say there is 99% confidence that the expected weekly loss for investing in the NASDAQ stock index and New York stock index will not exceed 2.01% and 1.72% respectively. Meanwhile, the VaR estimates also show that (VaR of 0.0172) it is less risky to invest in the New York stock index (VaR of 0.0172) than the NASDAQ stock index (VaR of 0.0201).

Table 5. Quantile-based VaR estimates of distributions for NASDAQ and New York stock indexes.

	NASDAQ stock index				
Distribution	**Quantile VaR estimates at different confident levels**				
	99% confidence level	**98% confidence level**	**97% confidence level**	**96% confidence level**	**95% confidence level**
APTL	0.0201	0.0248	0.0288	0.0324	0.0358
Logistic	0.0305	0.0261	0.0235	0.0216	0.0201
Normal	0.0298	0.0265	0.0243	0.0227	0.0214
Cauchy	0.1899	0.0960	0.0646	0.0489	0.0395

	New York Stock index				
Distribution	**Quantile VaR estimates at different confident levels**				
	99% confidence level	**98% confidence level**	**97% confidence level**	**96% confidence level**	**95% confidence level**
APTL	0.0172	0.0211	0.0244	0.0275	0.0303
Logistic	0.0275	0.0234	0.0210	0.0192	0.0179
Normal	0.0282	0.0249	0.0229	0.0213	0.0200
Cauchy	0.1540	0.0776	0.0521	0.0393	0.0316

8. CONCLUSIONS

In this research, we introduced and studied the alpha power transformed logistic distribution. Some mathematical properties of the APTL distribution are investigated. Estimation of the distribution parameters is done using the ML method of estimation and a simulation study is also performed to investigate the effectiveness of the estimates. Two portfolio stock indices, the NASDAQ and New York stock indices are used to show the flexibility of the APTL model, and the results show it is a better alternative than some familiar distributions like the Cauchy distribution, logistic distribution, Kumaraswamy logistic distribution, odd Lindley logistic distribution, and normal distributions used in literature to model stock portfolio data. The Value-at-risk estimates obtained also showed that the APTL distribution VaR estimates give lower expected maximum possible loss for both the NASDAQ and New York stock indices at higher confidence levels than those of the normal, logistic, and Cauchy distributions. It was also concluded, based on the VaR estimates, that it is less risky to invest in the New York stock index than the NASDAQ stock index.

REFERENCES

[1] F. Oladapo, A. P. Olufemi, S. O. Abayoni, and O. S. Daniel, "Exchange Rate Volatility and Stock Market Performance in Nigeria," *Nigerian Journal of Management Sciences,* vol. 6, no. 1, pp. 308-317, 2021.

[2] R. Bhowmik and S. Wang, "Stock Market Volatility and Return Analysis: A Systematic Literature Review," *Entropy*, vol. 22*, pp.* 1-18, 2020.

[3] A. Megaritis, N. Vlastakis and A. Triantafyllon, "Stock Market Volatility and Jumps in Times of Uncertainty," *Journal of International Money and Finance,* vol. 113, pp 1-24, 2021.

[4] E. Entri, W. Aipama, A. Razak, L. Sari, and S. Renil, "Stock Price Volatility During the COVID-19 Pandemic: The GARCH model," *Investment Management and Financial Innovations,* vol. 18, no. 3, pp 12-20, 2021.

[5] M. C. Korkmaz, E. Altun, H. M. Yousef, A. Z. Afify and S. Najarajah, "The BurrXPareto Distribution: Properties, Applications and VaR Estimation," *Journal of Risk and Financial Management,* vol. 11, no. 1, pp 1-16, 2018.
[6] G. Rodriguez, "Extreme value theory: An Application to the Peruvian Stock Market Returns," *Revista de Metodos Cuantitatuos Para la Economica y La Empressa, vol.* 23, pp 48-74 ,2017.
[7] M. I. Ahmad, "Analysis of Financial Data by Generalized Logistic Distribution," *International Journal of Statistics and applied mathematics, v*ol. 3, no. 5*, pp* 69-72, 2020.
[8] Y. An and E.A. Duah, "The Distribution Analysis of Extreme Returns of Nikkei 225 Index: Based on Extreme Value Distributions if GEV and GL," *Archives of Business Research, v*ol. 5, no. 1, pp 171-181, 2017.
[9] R. O. Gupta and D. Kundu, "Generalize Logistic Distributions," *Journal of Applied Statistical Sciences,* vol. 18, no. 1, pp 1-23, 2010.
[10] P. J. Hazarika and S. Chakraborty, "A Note on Alpha-Skewed-Generalized Logistic Distribution," *IOSR Journal of Mathematics*, vol. 11, no. 1, pp 85-88, 2015.
[11] L. P. Sapkota, "Exponentiated Exponential Logistic Distribution: Some Properties and Application," *Janapriya Journal of Interdisciplinary Studies,* pp 100-108, 2020.
[12] A. Mahdavi and D. Kundu, "A New Method for Generating Distributions with Application to Exponential Distribution," *Communications in Statistics-Theory and Methods*, vol 46, pp 6543-6557, 2017.
[13] K. M. Sakthivvel and A, Arul-chezhian, "Alpha Power Transformed Pareto Distribution and Its Properties with Application," *Malaya Journal of Matematik*, vol. 5, no. 1, pp 52-57, 2020.
[14] L. A. Rosa, S. Nurrohmah and I. Fithriani, "Alpha power transformed Lindsley distribution," *Proceeding of ICSA*, pp 130-142, 2019.
[15] I. Elbatal, M. Elgarhy and B. M. Golam-Kibria, "Alpha Power Transformed Weibull-G Family of Distributions and Applications," *Journal of Statistical Theory and Applications*. Vol. 20, no. 2, pp 340-354, 2020.
[16] R. A. ZeinEldin, M. A. Haq, S. Haghmi and M. Elsehaty, "Alpha Power Transformed Inverse Lomax Distribution with Different Methods of Estimation and Application," *Hindawi,* ID 1860813, pp 1-15, 2020.
[17] M. A. Aldahlan, "Alpha Power Transformed Log-Logistic Distribution with Application to Breaking Stress Data," *Advances of Mathematical Physics,* ID 2193787, pp 1-9, 2020.
[18] K. Syuhada, R. Nuraini and D. Mahfudhotin, "Quantile-Based Estimation VaR Forecast and Dependence Measure: A Simulation Approach," *Journal of applied mathematics*, ID 8276019, pp 1-14, 2020.
[19] D. L. Oslon and D. Wu, "The Impact of Distribution on Value-at-Risk Measures," *Mathematical and Computer Modelling*, Vol. 58, pp 1670-1676, 2013.
[20] D. Lindholm, "On Value-at-Risk and More Extreme A Study on Quantitative Market Risk Measurement," *Masters Thesis,* Statistics department, Uppsale Universit, 2015.
[21] I. Khindanova, S. Rachev, and E. Schwartz, "Stable Modelling of Value at Risk," *Mathematical and Computer Modelling*, vol. 34, pp 1223-1259, 2001.
[22] MathWorks. "Information criteria for model selection - MATLAB and Simulink" https://www.mathworks.com. *The MathWorks, Inc.* 1994-2023.
[23] "Logistic distribution." Wikipedia, 28 March 2023, wikipedia.org/wiki/ Logistic distribution.