InPrime: Indonesian Journal of Pure and Applied Mathematics
Vol. 5, No. 1 (2023), pp. 60 – 71, doi: 10.15408/inprime.v5i1.29133
p-ISSN 2686-5335, e-ISSN: 2716-2478

The Strong 3-Rainbow Index of Graphs Containing Three Cycles

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Abstract

The concept of a strong k-rainbow index is a generalization of a strong rainbow connection number, which has an interesting application in security systems in a communication network. Let G be an edge-colored connected graph of order n, where adjacent edges may be colored the same. A rainbow tree in G is a tree whose edges have distinct colors. For an integer k with $2 \leq k \leq n$, the strong k-rainbow index $srx_k(G)$ of G is the minimum number of colors needed to color all edges of G so that every k vertices of G are connected by a rainbow tree of minimum size. We focus on $k = 3$. It is clear that $srx_3(G) \leq \|G\|$, where the upper bound is sharp since the srx_3 of a tree equals its size. Hence, we are interested in studying how the srx_3 of a tree changes if we add some edges connecting two nonadjacent vertices in the tree. This paper is focused on graphs containing three cycles. We first determine a sharp upper bound of the srx_3 of graphs containing exactly three edge-disjoint cycles. We also determine the exact values of srx_3 of theta graph $\theta(a_1, a_2, a_3)$ for certain values of a_1, a_2, and a_3.
Keywords: cycle; rainbow coloring; rainbow Steiner tree; theta graph; tree.

Abstrak

Konsep indeks pelangi-k kuat merupakan perumuman dari bilangan terhubung pelangi kuat yang memiliki aplikasi menarik dalam sistem keamanan jaringan komunikasi. Misalkan G adalah suatu graf terhubung berorde n yang memiliki suatu pewarnaan sisi, dimana dua sisi bertetangga boleh memiliki warna yang sama. Pohon pelangi di G adalah pohon yang setiap sisinya memiliki warna berbeda. Untuk suatu bilangan bulat k dengan $2 \leq k \leq n$, indeks pelangi-k kuat $srx_k(G)$ graf G adalah banyak warna minimum yang dibutuhkan untuk mewarnai semua sisi di G sehingga setiap k titik di G dihubungkan oleh suatu pohon pelangi berukuran minimum. Kami fokus pada $k = 3$. Jelas bahwa $srx_3(G) \leq \|G\|$, dimana batas atas ini merupakan batas ketat karena srx_3 pohon sama dengan ukurannya. Karena itu, kami tertarik untuk mempelajari bagaimana srx_3 pohon berubah jika ditambahkan beberapa sisi yang menghubungkan dua titik tidak bertetangga di pohon tersebut. Artikel ini difokuskan pada graf yang memuat tiga siklus. Pertama, kami menentukan batas atas ketat srx_3 graf yang memuat tepat tiga siklus saling lepas sisi. Kami juga menentukan nilai eksak srx_3 graf theta $\theta(a_1, a_2, a_3)$ untuk beberapa nilai a_1, a_2, dan a_3 tertentu.
Kata Kunci: *siklus; pewarnaan pelangi; pohon Steiner pelangi; graf theta; pohon.*

2020MSC: 05C05, 05C15, 05C38, 05C40.

1. INTRODUCTION

All graphs considered in this paper are nontrivial, simple, and connected. We follow the notation and terminology of Diestel [1] unless otherwise stated. Chartrand et al. in 2008 introduced the concept of the rainbow connection number of a graph [2]. This concept has an interesting application in a security system in a communication network, which can be modeled by graph G. To obtain a secure

Submitted November 19th, 2022, Revised May 25th, 2023, Accepted for publication May 29th, 2023.

communication network, we can assign some passwords to an information transfer line between people, which may have other people as intermediaries so that no password is repeated. The minimum number of these passwords is represented by the rainbow connection number $rc(G)$ of G. In the same paper, Chartrand et al. also introduced the concept of a strong rainbow connection number $src(G)$ of G, which represents the minimum number of passwords needed in a communication network so that people can transfer information securely and quickly.

Computing the rc of graphs is an NP-Hard problem [3], [4]. Therefore, many previous researchers studied the rc of graphs by limiting their study to certain classes of graphs, e.g. [2], [5], [6], [7], [8], [9], [10], [11]. We refer the readers to [12], [13] for some detailed surveys on rc of graphs.

Later, Awanis and Salman introduced the concept of a strong k-rainbow index of a graph [14], which is a generalization of the strong rainbow connection number of a graph. Let G be an edge-colored connected graph of order n, where adjacent edges may be colored the same. A tree T in G is a *rainbow tree* if no two edges of T are colored the same. Let S be a set of vertices of G. The minimum size of a tree connecting S in G is called the *Steiner distance* $d(S)$ of S. Such a tree is called a *Steiner* S*-tree*. The maximum Steiner distance of S among all sets S of k vertices of G is called the k*-Steiner diameter* $sdiam_k(G)$ of G [15]. If $S = \{u, v\}$, then $d(u, v)$ is the *distance* between u and v in G, and $sdiam_2(G)$ is the *diameter* $diam(G)$ of G. Furthermore, the Steiner $\{u, v\}$-tree is called the $u - v$ *geodesic* [2]. A *strong* k*-rainbow coloring* of G is an edge-coloring of G having the property that for every set S of k vertices of G, there exists a rainbow Steiner S-tree in G. The minimum number of colors needed in a strong k-rainbow coloring of G is called the *strong* k*-rainbow index* $srx_k(G)$ of G. If $k = 2$, then $srx_2(G)$ is the *strong rainbow connection number* $src(G)$ of G [2]. It follows that for every connected graph G of order n, $src(G) = srx_2(G) \leq srx_3(G) \leq \cdots \leq srx_n(G)$ [14].

It is clear that $sdiam_k(G)$ is the natural lower bound for $srx_k(G)$. Furthermore, since every edge-coloring of G that assigns distinct colors to all edges of G is a strong k-rainbow coloring, Awanis, and Salman in [14] showed that

$$sdiam_k(G) \leq srx_k(G) \leq \|G\|, \tag{1}$$

where $\|G\|$ denotes the size of G. They also provided the exact values of srx_3 of some certain graphs and their amalgamation. Some of these results are given in Observation 1.1 and Theorems 1.2 and 1.3. Observation 1.1 shows how to color the bridges of a connected graph. An edge e of a connected graph G is called a *bridge* of G if $G - e$ is disconnected. Meanwhile, Theorems 1.2 and 1.3 provide the exact values of srx_3 of trees and cycles, respectively. There are also some results about the srx_3 of graphs formed by other graph operations, such as edge amalgamation of graphs [16], comb product of graphs [17], and edge-comb product of graphs [18]. Some authors are also interested to study the characterization of graphs G with $srx_3(G) = 2$ [19].

Observation 1.1. [14] Let G be a connected graph of order $n \geq 3$ and c be a strong 3-rainbow coloring of G. If e and f are two distinct bridges of G, then $c(e) \neq c(f)$.

Theorem 1.2. [14] For each integer $n \geq 3$, $srx_3(T_n) = \|T_n\| = n - 1$.

Theorem 1.3. [14] For each integer $n \geq 3$, the strong 3-rainbow index of C_n is

$$srx_3(C_n) = \begin{cases} 2, & \text{for } n = 3; \\ n-2, & \text{for } 4 \leq n \leq 6 \text{ or } n = 8; \\ n, & \text{for } n = 7 \text{ or } n \geq 9. \end{cases}$$

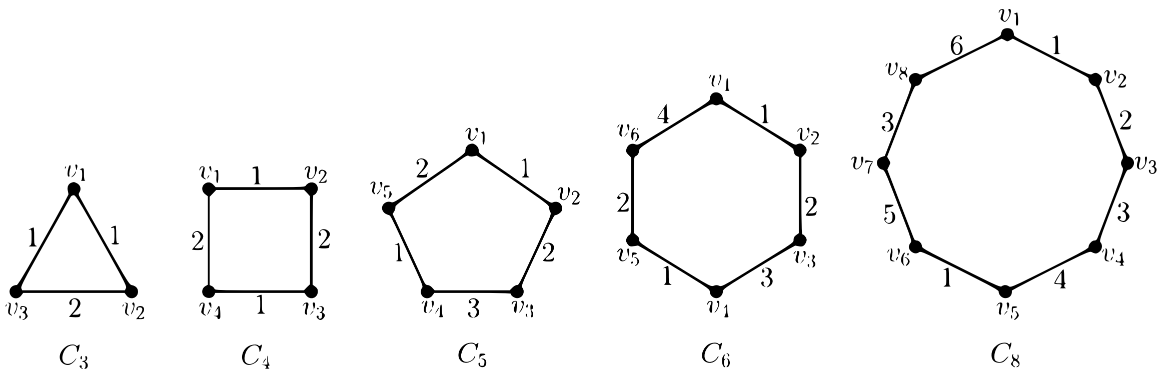

Figure 1. Strong 3-rainbow colorings of C_3, C_4, C_5, C_6, and C_8.

Following Theorem 1.2, we are interested in studying how the srx_3 of a tree changes if we add some edges connecting two nonadjacent vertices in the tree. This paper is focused on graphs containing three cycles. We first determine an upper bound of the srx_3 of graphs containing exactly three edge-disjoint cycles, then we prove the sharpness of the upper bound by providing a graph containing exactly three edge-disjoint cycles whose srx_3 equals the upper bound. We also determine the exact values of srx_3 of theta graph $\theta(a_1, a_2, a_3)$ for certain values of a_1, a_2, and a_3.

2. RESULTS AND DISCUSSIONS

For simplifying, we define some notations as follows. We define $[a, b]$ as a set of all integers x with $a \leq x \leq b$, $T = \{e_1, e_2, \dots, e_n\}$ as a tree with edge set $\{e_1, e_2, \dots, e_n\}$, and $c(X)$ as a set of colors assigned to the edges in $X \subseteq E(G)$.

For $n \geq 7$ and $g_1, g_2, g_3 \geq 3$, let H be a connected graph of order n containing exactly three edge-disjoint cycles C_{g_1}, C_{g_2}, and C_{g_3}. Hence, there exist exactly two paths P_1 and P_2 connecting certain two cycles as given in Figure 2. Without loss of generality, let $V(C_{g_i}) = \{v_i^1, v_i^2, \dots, v_i^{g_i}\}$ and $E(C_{g_i}) = \{v_i^p v_i^{p+1} : p \in [1, g_i], v_i^{g_i+1} = v_i^1\}$ for each $i \in [1,3]$ such that $P_1 \coloneqq v_1^1 - v_2^1$ and $P_2 \coloneqq v_2^t - v_3^1$ for some $t \in [1, g_2]$. Observe that if $t = 1$, then $|E(P_1) \cap E(P_2)| \geq 0$.

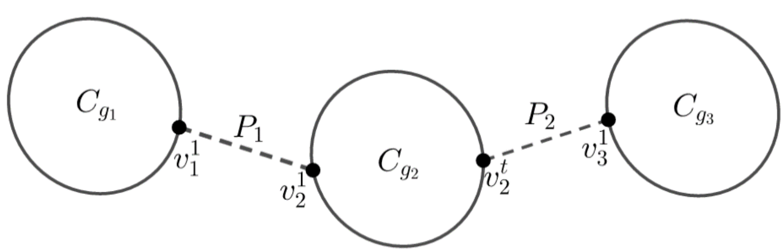

Figure 2. A graph containing exactly three cycles.

We first determine an upper bound for $srx_3(H)$ as given in the following theorem.

Theorem 2.1. Let H be a connected graph of order $n \geq 7$ containing exactly three edge-disjoint cycles of length at least 3. Then $srx_3(H) \leq \|H\| - 2$.

Proof. We show $srx_3(H) \leq \|H\| - 2$ by defining a strong 3-rainbow coloring c of H as follows.

1. Assign colors $1,2,\dots,g_2$ to the edges of C_{g_2}.
2. Define $c\left(v_1^{\lfloor\frac{g_1}{2}\rfloor+1}v_1^{\lfloor\frac{g_1}{2}\rfloor+2}\right) = c\left(v_2^{\lfloor\frac{g_2}{2}\rfloor+1}v_2^{\lfloor\frac{g_2}{2}\rfloor+2}\right)$ and $c\left(v_3^{\lfloor\frac{g_3}{2}\rfloor+1}v_3^{\lfloor\frac{g_3}{2}\rfloor+2}\right) = c\left(v_2^{\lfloor\frac{g_2}{2}\rfloor+t}v_2^{\lfloor\frac{g_2}{2}\rfloor+t+1}\right)$.
3. Assign colors $g_2+1, g_2+2, \dots, \|H\|-2$ to the remaining $\|H\| - g_2 - 2$ edges of H.

Observe that the edge coloring above provides the following properties.

- For each $i \in [1,3]$, all edges of C_{g_i} have distinct colors.
- For distinct $i,j \in \{1,2\}$ or $i,j \in \{1,3\}$, $p \in [1,g_i]$, and $q,r \in [1,g_j]$, there exists a rainbow $v_i^1 - v_i^p$ geodesic T_i in C_{g_i} and a rainbow Steiner $\{v_j^1, v_j^q, v_j^r\}$-tree T_j in C_{g_j} such that $c(E(T_i)) \cap c\left(E(T_j)\right) = \emptyset$.
- For $i = 2$ and $j = 3$, we have two properties as follows.
  - There exist a rainbow $v_2^t - v_2^p$ geodesic T_1 in C_{g_2} and a rainbow Steiner $\{v_3^1, v_3^q, v_3^r\}$-tree T_2 in C_{g_3} for each $p \in [1,g_2]$ and $q,r \in [1,g_3]$ such that $c(E(T_1)) \cap c(E(T_2)) = \emptyset$.
  - There exist a rainbow Steiner $\{v_2^t, v_2^p, v_2^q\}$-tree T_1 in C_{g_2} and a rainbow $v_3^1 - v_3^r$ geodesic T_2 in C_{g_3} for each $p,q \in [1,g_2]$ and $r \in [1,g_3]$ such that $c(E(T_1)) \cap c(E(T_2)) = \emptyset$.
- If H contains bridges, then every bridge of H is colored with distinct colors which are not used for $E(C_{g_i})$ for all $i \in [1,3]$.

By using the properties above repeatedly, we can find a rainbow Steiner tree connecting every three vertices of H. Thus, the theorem holds. ■

The upper bound given in Theorem 2.1 is sharp. Theorem 2.3 shows that there exists a connected graph H containing exactly three edge-disjoint cycles of a certain length with $srx_3(H) = \|H\| - 2$. We first need the following lemma.

Lemma 2.2. For $n \geq 4$ and $g \geq 3$, let G be a connected graph of order n containing a cycle C_g. Let c be a strong 3-rainbow coloring of G. If $e \in E(C_g)$ and f is an arbitrary bridge of G, then $c(e) \neq c(f)$.

Proof. Let $V(C_g) = \{v_1, v_2, \dots, v_g\}$ and $E(C_g) = \{v_i v_{i+1} : i \in [1,g], v_{g+1} = v_1\}$. Let $e = v_i v_{i+1}$ for $i \in [1,g]$ be an arbitrary edge of C_g and $f = xy$ be an arbitrary bridge of G. Assume that $d(C_g, x) < d(C_g, y)$. Thus, by considering $\{v_i, v_{i+i}, y\}$, we obtain that $c(e) \neq c(f)$. ■

Theorem 2.3. Let g_1, g_2, and g_3 be three integers equal to 7 or at least 9. Let H be a connected graph containing exactly three edge-disjoint cycles of length g_1, g_2, and g_3. If g_1 and g_3 are odd or have distinct parity, then $srx_3(H) = \|H\| - 2$.

Proof. Let C_{g_1}, C_{g_2}, and C_{g_3} be the three edge-disjoint cycles contained in H. It follows by Theorem 2.1 that $srx_3(H) \leq \|H\| - 2$. To prove the lower bound, suppose that $srx_3(H) \leq \|H\| - 3$. Let $c: E(H) \to [1, \|H\| - 3]$ be a strong 3-rainbow coloring of H. Let X be a set of colors assigned to the bridges of H and Y be a set of colors assigned to the edges of C_{g_i} for all $i \in [1,3]$. Since H contains $\|H\| - g_1 - g_2 - g_3$ bridges, we have $|X| \geq \|H\| - g_1 - g_2 - g_3$ by Observation 1.1. Note that by Lemma 2.2, all edges of C_{g_i} for each $i \in [1,3]$ should have distinct colors from X. Thus, $X \cap Y = \emptyset$, which implies $|Y| \leq g_1 + g_2 + g_3 - 3$. Without loss of generality, let $Y = [1, g_1 + g_2 + g_3 - 3]$. Since $srx_3(C_{g_2}) \geq g_2$ by Theorem 1.3, without loss of generality, let $c\left(E\left(C_{g_2}\right)\right) = [1, g_2]$. Now, consider cycles C_{g_1} and C_{g_3}. For each $i \in \{1,3\}$, let $A_i = E\left(C_{g_i}\right) \setminus \left\{v_i^{\left\lfloor\frac{g_i}{2}\right\rfloor+1} v_i^{\left\lfloor\frac{g_i}{2}\right\rfloor+2}\right\}$ if g_i is odd or $A_i = E\left(C_{g_i}\right) \setminus \left\{v_i^{\frac{g_i}{2}} v_i^{\frac{g_i}{2}+1}, v_i^{\frac{g_i}{2}+1} v_i^{\frac{g_i}{2}+2}\right\}$ if g_i is even. If g_1 and g_3 are odd, then by considering $\{v_1^1, v_1^p, v_3^q\}$ for $p \in \left\{\left\lfloor\frac{g_1}{2}\right\rfloor + 1, \left\lfloor\frac{g_1}{2}\right\rfloor + 2\right\}$ and $q \in \left\{\left\lfloor\frac{g_3}{2}\right\rfloor + 1, \left\lfloor\frac{g_3}{2}\right\rfloor + 2\right\}$, we obtain that $c(A_1) \cap c(A_3) = \emptyset$. A similar argument applies if g_1 and g_3 have distinct parity. Thus,

$$c(A_1) \cap c(A_3) = \emptyset. \tag{2}$$

Observe that every edge $e = xy \in E\left(C_{g_2}\right)$ should be contained in any rainbow Steiner $\{x, y, v_i^p\}$-tree for each $i \in \{1,3\}$ and $p \in \left\{\left\lfloor\frac{g_i}{2}\right\rfloor + 1, \left\lfloor\frac{g_i}{2}\right\rfloor + 2\right\}$ if g_i is odd or $p \in \left\{\frac{g_i}{2}, \frac{g_i}{2} + 2\right\}$ if g_i is even. This implies $c(A_i) \nsubseteq [1, g_2]$ for each $i \in \{1,3\}$. Now, we consider two cases as follows.

Case 1. g_1 and g_3 are odd

Since every three vertices of C_{g_i} for each $i \in \{1,3\}$ are connected by a rainbow Steiner tree contained in C_{g_i}, we have $|c(A_i)| \geq g_i - 1$ by Theorem 1.3. It follows by equation (2) that $|c(A_1) \cup c(A_3)| \geq g_1 + g_3 - 2$, which is impossible since there are at most $g_1 + g_3 - 3$ colors left from Y that have not been used.

Case 2. g_1 and g_3 have distinct parity

Without loss of generality, let g_1 be even and g_3 be odd. By using a similar argument as Case 1, we have $|c(A_1)| \geq g_1 - 2$ and $|c(A_3)| \geq g_3 - 1$ by Theorem 1.3. It follows by equation (2) that $|c(A_1) \cup c(A_3)| \geq g_1 + g_3 - 3$, which implies $c(A_1) \cup c(A_3) = [g_2 + 1, g_2 + 2, \dots, g_1 + g_2 + g_3 - 3]$. It means we have used all colors from Y. Now, consider edges $v_1^{\frac{g_1}{2}} v_1^{\frac{g_1}{2}+1}$ and $v_1^{\frac{g_1}{2}+1} v_1^{\frac{g_1}{2}+2}$. By using Theorem 1.3 and considering $\left\{v_1^{\frac{g_1}{2}}, v_1^{\frac{g_1}{2}+2}, v_3^p\right\}$ for $p \in \left\{\left\lfloor\frac{g_3}{2}\right\rfloor + 1, \left\lfloor\frac{g_3}{2}\right\rfloor + 2\right\}$, we have $\left\{c\left(v_1^{\frac{g_1}{2}} v_1^{\frac{g_1}{2}+1}\right), c\left(v_1^{\frac{g_1}{2}+1} v_1^{\frac{g_1}{2}+2}\right)\right\} \nsubseteq c(A_1) \cup c(A_3)$. This implies $\left\{c\left(v_1^{\frac{g_1}{2}} v_1^{\frac{g_1}{2}+1}\right), c\left(v_1^{\frac{g_1}{2}+1} v_1^{\frac{g_1}{2}+2}\right)\right\} \subseteq [1, g_2]$. If g_2 is even, then by considering $\left\{v_1^{\frac{g_1}{2}}, v_1^{\frac{g_1}{2}+2}, v_2^p\right\}$ for $p \in \left\{\frac{g_2}{2}, \frac{g_2}{2} + 2\right\}$, we have $\left\{c\left(v_1^{\frac{g_1}{2}} v_1^{\frac{g_1}{2}+1}\right), c\left(v_1^{\frac{g_1}{2}+1} v_1^{\frac{g_1}{2}+2}\right)\right\} = \left\{c\left(v_2^{\frac{g_2}{2}} v_2^{\frac{g_2}{2}+1}\right), c\left(v_2^{\frac{g_2}{2}+1} v_2^{\frac{g_2}{2}+2}\right)\right\}$. However, there is no rainbow Steiner $\left\{v_1^{\frac{g_1}{2}}, v_1^{\frac{g_1}{2}+2}, v_2^{\frac{g_2}{2}+1}\right\}$-tree, a contradiction. If g_2 is odd, then by considering $\left\{v_1^{\frac{g_1}{2}}, v_1^{\frac{g_1}{2}+2}, v_2^{p}\right\}$ for $p \in \left\{\left\lfloor\frac{g_2}{2}\right\rfloor+1, \left\lfloor\frac{g_2}{2}\right\rfloor+2\right\}$, we have $\left\{c\left(v_1^{\frac{g_1}{2}} v_1^{\frac{g_1}{2}+1}\right), c\left(v_1^{\frac{g_1}{2}+1} v_1^{\frac{g_1}{2}+2}\right)\right\} = \left\{c\left(v_2^{\left\lfloor\frac{g_2}{2}\right\rfloor+1} v_2^{\left\lfloor\frac{g_2}{2}\right\rfloor+2}\right)\right\}$, which is impossible since two adjacent edges should have distinct colors. ■

The illustration of a strong 3-rainbow coloring of H with $g_1 = g_2 = 7$ and $g_3 = 9$ is given in Figure 3.

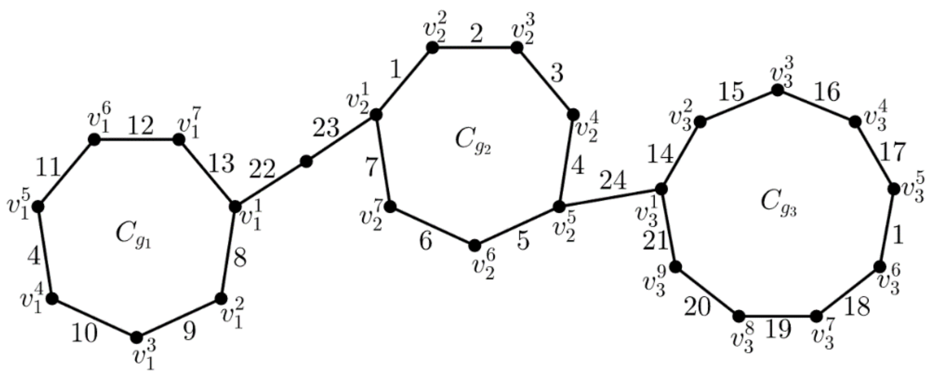

Figure 3. A strong 3-rainbow coloring of H with $g_1 = g_2 = 7$ and $g_3 = 9$.

Another class of graphs containing exactly three cycles is a theta graph. For $a_3 \geq a_2 \geq a_1 \geq 2$, a *theta graph* $\theta(a_1, a_2, a_3)$ is a graph constructed by three internally disjoint paths of length a_1, a_2, and a_3 which have the same end vertices. For distinct $i, j \in [1,3]$, let $P_{a_i} := x v_i^1 v_i^2 \ldots v_i^{a_i-1} y$ be a path of length a_i contained in $\theta(a_1, a_2, a_3)$, and $C_{a_i+a_j} := P_{a_i} \cup P_{a_j}$. Theorems 2.6 and 2.7 provide the $srx_3(\theta(a_1, a_2, a_3))$ for certain values of a_1, a_2, and a_3. We first need the following two observations.

Observation 2.4. For $a_3 \geq a_2 \geq a_1 \geq 2$, let $\theta(a_1, a_2, a_3)$ be a theta graph of order $a_1 + a_2 + a_3 - 1$ which admits a strong 3-rainbow coloring. If $a_1 = a_2 = a_3$, then any rainbow Steiner tree connecting every three vertices of $C_{a_i+a_j}$ should be contained in $C_{a_i+a_j}$ for distinct $i, j \in [1,3]$.

Proof. For distinct $i, j \in [1,3]$, let S be a set of three vertices of $C_{a_i+a_j}$. We consider three cases as follows.

Case 1. $\{x, y\} \subseteq S$

Either P_{a_i} or P_{a_j} is a rainbow Steiner S-tree.

Case 2. $x \in S$ or $y \in S$

Without loss of generality, let $x \in S$. If $S = \{x, v_i^p, v_i^q\}$ for distinct $p, q \in [1, a_i - 1]$, then the rainbow Steiner S-tree is contained in P_{a_i}. A similar argument applies if $S = \{x, v_j^p, v_j^q\}$ for distinct $p, q \in [1, a_j - 1]$. Now, if $S = \{x, v_i^p, v_j^q\}$ for $p \in [1, a_i - 1]$ and $q \in [1, a_j - 1]$, let S' be the Steiner distance of S contained in $C_{a_i+a_j}$. Observe that any rainbow Steiner S-tree that passes through

P_{a_k} for $k \neq i$ and $k \neq j$ has a size at least $S' + 1$. Thus, it is clear that the rainbow Steiner S-tree is contained in $C_{a_i+a_j}$.

Case 3. $\{x, y\} \nsubseteq S$

Without loss of generality, let $S = \{v_i^p, v_i^q, v_j^r\}$ for distinct $p, q \in [1, a_i - 1]$ and $r \in [1, a_j - 1]$. By using a similar argument as Case 2, we will obtain that any rainbow Steiner S-tree is contained in $C_{a_i+a_j}$. ■

Observation 2.5. For $a_3 \geq a_2 \geq a_1 \geq 2$, let $\theta(a_1, a_2, a_3)$ be a theta graph of order $a_1 + a_2 + a_3 - 1$ which admits a strong 3-rainbow coloring. If $a_1 = a_2$ and $a_3 > a_1$, then any rainbow Steiner tree connecting every three vertices of $C_{a_i+a_j}$ should be contained in $C_{a_i+a_j}$ for distinct $i, j \in [1,3]$.

Proof. For distinct $i, j \in [1,3]$, let S be a set of three vertices of $C_{a_i+a_j}$. If $S \subseteq V(C_{a_1+a_2})$, then by Observation 2.4, any rainbow Steiner S-tree is contained in $C_{a_1+a_2}$. Hence, we assume that $S \subseteq V(C_{a_i+a_3})$ for $i \in \{1,2\}$. By considering the three cases and using a similar argument as Observation 2.4, we will obtain that any rainbow Steiner S-tree is contained in $C_{a_i+a_3}$. ■

Theorem 2.6. For $a_3 \geq a_2 \geq a_1 \geq 2$, let $\theta(a_1, a_2, a_3)$ be a theta graph of order $a_1 + a_2 + a_3 - 1$. If $a_1 = a_2 = a_3$, then

$$srx_3(\theta(a_1, a_1, a_1)) = \begin{cases} 3, & \text{for } a_1 = 2; \\ 4, & \text{for } a_1 = 3; \\ 8, & \text{for } a_1 = 4; \\ 3a_1, & \text{for } a_1 \geq 5. \end{cases}$$

Proof. We consider four cases as follows.

Case 1. $a_1 = 2$

Since any rainbow Steiner $\{v_1^1, v_2^1, v_3^1\}$-tree has a size of at least 3, then $srx_3(\theta(2,2,2)) \geq 3$. To prove the upper bound, we define a strong 3-rainbow coloring of $\theta(2,2,2)$ as given in Figure 4.

Case 2. $a_1 = 3$

Since any rainbow Steiner $\{v_1^1, v_2^2, v_3^1\}$-tree has a size of at least 4, then $srx_3(\theta(3,3,3)) \geq 4$. To prove the upper bound, we define a strong 3-rainbow coloring of $\theta(3,3,3)$ as given in Figure 4.

Case 3. $a_1 = 4$

Suppose that $srx_3(\theta(4,4,4)) \leq 7$. Let $c: E(\theta(4,4,4)) \to [1,7]$ be a strong 3-rainbow coloring of $\theta(4,4,4)$. Since there are two possible rainbow Steiner $\{v_1^2, v_2^2, v_3^2\}$-trees, without loss of generality, let $T = \{xv_1^1, v_1^1v_1^2, xv_2^1, v_2^1v_2^2, xv_3^1, v_3^1v_3^2\}$ be the rainbow Steiner $\{v_1^2, v_2^2, v_3^2\}$-tree with $c(vv_i^1) = i$ and $c(v_i^1v_i^2) = i + 3$ for each $i \in [1,3]$. Now, by considering $\{v_i^1, v_i^3, v_j^1\}$, $\{x, y, v_i^2\}$, and $\{v_i^3, v_j^1, v_j^3\}$ for distinct $i, j \in [1,3]$, we have

$$c(v_1^2v_1^3) \in \{5,6,7\}, c(v_1^3y) \in \{2,3,7\}, c(v_2^2v_2^3) \in \{4,6,7\}, c(v_2^3y) \in \{1,3,7\},$$
$$c(v_3^2v_3^3) \in \{4,5,7\}, \text{ and } c(v_3^3y) \in \{1,2,7\}. \tag{3}$$

Next, consider $\{y, v_i^2, v_j^2\}$ for distinct $i, j \in [1,3]$. Since $T = \{v_i^2 v_i^3, v_i^3 y, v_j^2 v_j^3, v_j^3 y\}$ is the only possible rainbow Steiner tree connecting these three vertices, we obtain that edges $v_1^2 v_1^3$, $v_1^3 y$, $v_2^2 v_2^3$, $v_2^3 y$, $v_3^2 v_3^3$, and $v_3^3 y$ should have distinct colors. Hence, we further consider two subcases as follows.

Subcase 3.1. There exists $i \in [1,3]$ such that either $c(v_i^2 v_i^3) = 7$ or $c(v_i^3 y) = 7$

Without loss of generality, let $i = 1$. If $c(v_1^2 v_1^3) = 7$, this forces $c(v_1^3 y) \in \{2,3\}$, $c(v_2^2 v_2^3) \in \{4,6\}$, $c(v_2^3 y) \in \{1,3\}$, $c(v_3^2 v_3^3) \in \{4,5\}$, and $c(v_3^3 y) \in \{1,2\}$ by equation (3). Observe that by equation (3), there are two possible colorings of edges $v_1^3 y$, $v_2^3 y$, and $v_3^3 y$ as follows.

(i) $c(v_1^3 y) = 3$, $c(v_2^3 y) = 1$, and $c(v_3^3 y) = 2$
If $c(v_2^2 v_2^3) = 4$, then there is no rainbow Steiner $\{x, v_1^2, v_2^3\}$-tree. Hence, we have $c(v_2^2 v_2^3) = 6$. A similar argument applies for edge $v_3^2 v_3^3$, thus $c(v_3^2 v_3^3) = 4$. However, there is no rainbow Steiner $\{v_1^1, v_2^3, v_3^2\}$-tree, a contradiction.

(ii) $c(v_1^3 y) = 2$, $c(v_2^3 y) = 3$, and $c(v_3^3 y) = 1$
By using a similar argument as Case (i), we will obtain that there is no rainbow Steiner $\{v_1^1, v_2^2, v_3^3\}$-tree, a contradiction.

Now, if $c(v_1^3 y) = 7$, then $c(v_1^2 v_1^3) \in \{5,6\}$, $c(v_2^2 v_2^3) \in \{4,6\}$, $c(v_2^3 y) \in \{1,3\}$, $c(v_3^2 v_3^3) \in \{4,5\}$, and $c(v_3^3 y) \in \{1,2\}$. Observe that by equation (3), there are two possible colorings of edges $v_1^2 v_1^3$, $v_2^2 v_2^3$, and $v_3^2 v_3^3$, which are $c(v_1^2 v_1^3) = 5$, $c(v_2^2 v_2^3) = 6$, and $c(v_3^2 v_3^3) = 4$, or $c(v_1^2 v_1^3) = 6$, $c(v_2^2 v_2^3) = 4$, and $c(v_3^2 v_3^3) = 5$. By using a similar argument as case $c(v_1^2 v_1^3) = 7$, we will obtain a contradiction.

Subcase 3.2. For each $i \in [1,3]$, $c(v_i^2 v_i^3) \neq 7$ and $c(v_i^3 y) \neq 7$

It follows by equation (3) that $c(v_1^2 v_1^3) \in \{5,6\}$, $c(v_1^3 y) \in \{2,3\}$, $c(v_2^2 v_2^3) \in \{4,6\}$, $c(v_2^3 y) \in \{1,3\}$, $c(v_3^2 v_3^3) \in \{4,5\}$, and $c(v_3^3 y) \in \{1,2\}$. For this case, there are also two possible colorings of edges $v_1^3 y$, $v_2^3 y$, and $v_3^3 y$ by equation (3), which are $c(v_1^3 y) = 3$, $c(v_2^3 y) = 1$, and $c(v_3^3 y) = 2$, or $c(v_1^3 y) = 2$, $c(v_2^3 y) = 3$, and $c(v_3^3 y) = 1$. Hence, by using a similar argument as Subcase 3.1, we will obtain a contradiction.

Next, to prove the upper bound, we define a strong 3-rainbow coloring of $\theta(4,4,4)$ as given in Figure 4.

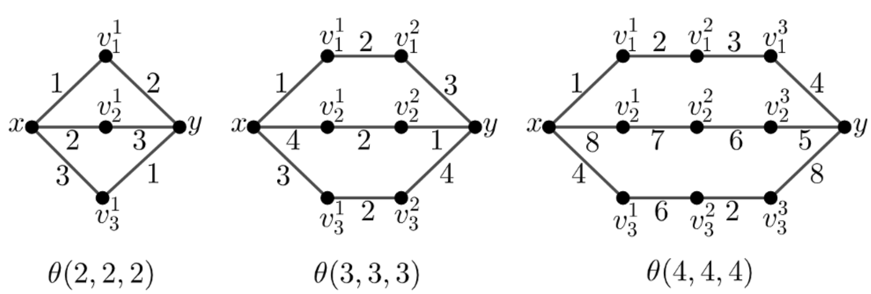

Figure 4. Strong 3-rainbow colorings of $\theta(2,2,2)$, $\theta(3,3,3)$, and $\theta(4,4,4)$.

Case 4. $a_1 \geq 5$

Let c be a strong 3-rainbow coloring of $\theta(a_1, a_1, a_1)$. By Theorem 1.3 and Observation 2.4, all edges of $\theta(a_1, a_1, a_1)$ should have distinct colors. Thus, $srx_3(\theta(a_1, a_1, a_1)) \geq 3a_1$. Furthermore, since $\|\theta(a_1, a_1, a_1)\| = 3a_1$, it follows by equation (1) that $srx_3(\theta(a_1, a_1, a_1)) = 3a_1$. ■

Theorem 2.7. For $a_3 \geq a_2 \geq a_1 \geq 2$, let $\theta(a_1, a_2, a_3)$ be a theta graph of order $a_1 + a_2 + a_3 - 1$. If $a_1 = a_2$ and $a_3 > a_1$, then

$$srx_3(\theta(a_1, a_1, a_3)) = \begin{cases} a_3, & \text{for } a_1 = 2 \text{ and } a_3 \in \{3,4\}; \\ 7, & \text{for } a_1 = 2 \text{ and } a_3 = 6, \text{or } a_1 = 3 \text{ and } a_3 = 5; \\ 2a_1 + a_3 - 2, & \text{for } a_1 = 2 \text{ and } a_3 = 5 \text{ } or \text{ } a_3 \geq 7, \text{or} \\ & a_1 = 3 \text{ and } a_3 = 4 \text{ } or \text{ } a_3 \geq 6, \text{or } a_1 = 4 \text{ and } a_3 \geq 5; \\ 2a_1 + a_3, & \text{for } a_1 \geq 5 \text{ and } a_3 \geq 6. \end{cases}$$

Proof. We consider four cases as follows.

Case 1. $a_1 = 2$ and $a_3 \in \{3,4\}$

Since $sdiam_3(\theta(2,2,a_3)) = a_3$, we have $srx_3(\theta(2,2,a_3)) \geq a_3$ by equation (1). To prove the upper bound, we define a strong 3-rainbow coloring of $\theta(2,2,a_3)$ as given in Figure 5.

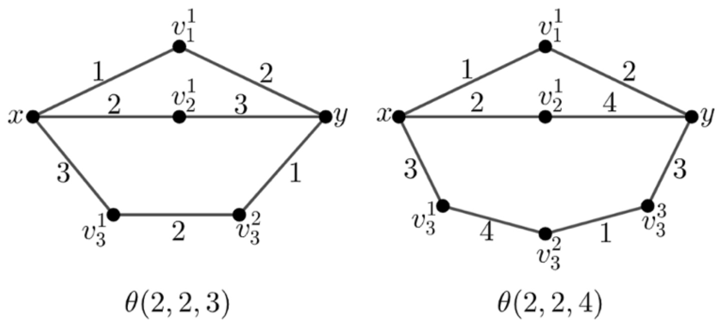

Figure 5. Strong 3-rainbow colorings of $\theta(2,2,3)$ and $\theta(2,2,4)$.

Case 2. $a_1 = 2$ and $a_3 = 6$, or $a_1 = 3$ and $a_3 = 5$

Suppose that $srx_3(\theta(a_1, a_1, a_3) \leq 6$. Let $c : E(\theta(a_1, a_1, a_3)) \to [1,6]$ be a strong 3-rainbow coloring of $\theta(a_1, a_1, a_3)$. Consider graph $C_{a_1+a_3}$. By Observation 2.5, any rainbow Steiner tree connecting every three vertices of $C_{a_1+a_3}$ should be contained in $C_{a_1+a_3}$. It follows Theorem 1.3 that we need at least 6 colors to color all edges of $C_{a_1+a_3}$.

For $a_1 = 2$ and $a_3 = 6$, there is only one possible edge-coloring of $C_{a_1+a_3}$. Without loss of generality, color all edges of $C_{a_1+a_3}$ as given in Figure 6(a). Now, consider edge v_2^1y. By considering $\{y, v_1^1, v_2^1\}$, $\{y, v_2^1, v_3^2\}$, and $\{y, v_2^1, v_3^3\}$, we obtain that $c(v_2^1y) \notin \{1,2,3,4,6\}$. These forces $c(v_2^1y) = 5$. However, there is no rainbow Steiner $\{v_2^1, v_3^2, v_3^5\}$-tree, a contradiction.

For $a_1 = 3$ and $a_3 = 5$, there are two possible edge colorings of $C_{a_1+a_3}$. Without loss of generality, color all edges of $C_{a_1+a_3}$ as given in Figures 6(b) and 6(c). First, consider Figure 6(b). By considering $\{x, v_1^2, v_2^1\}$, $\{v_2^1, v_3^1, v_3^3\}$, and $\{x, v_2^2, v_3^4\}$, we obtain that $c(xv_2^1) \notin [1,6]$. This implies we need one new distinct color to color edge xv_2^1, which is impossible. Now, consider Figure 6(c). First, consider edge $v_2^1v_2^2$. By considering $\{x, v_1^1, v_2^2\}$, $\{x, v_2^2, v_3^2\}$, and $\{y, v_2^1, v_3^3\}$, we obtain that $c(v_2^1v_2^2) \notin \{1,3,4,5,6\}$. This forces $c(v_2^1v_2^2) = 2$. By symmetry and using a similar argument as the previous case, we can show that $c(xv_2^1) \in \{3,5\}$ and $c(v_2^2y) \in \{1,6\}$. If $c(xv_2^1) = 5$, then there is no rainbow Steiner $\{x, v_2^2, v_3^3\}$-tree, a contradiction. A similar argument applies if $c(v_2^2y) = 6$. Thus, $c(xv_2^1) = 3$ and $c(v_2^2y) = 1$. However, there is no rainbow Steiner $\{x, v_1^2, v_2^2\}$-tree, a contradiction.

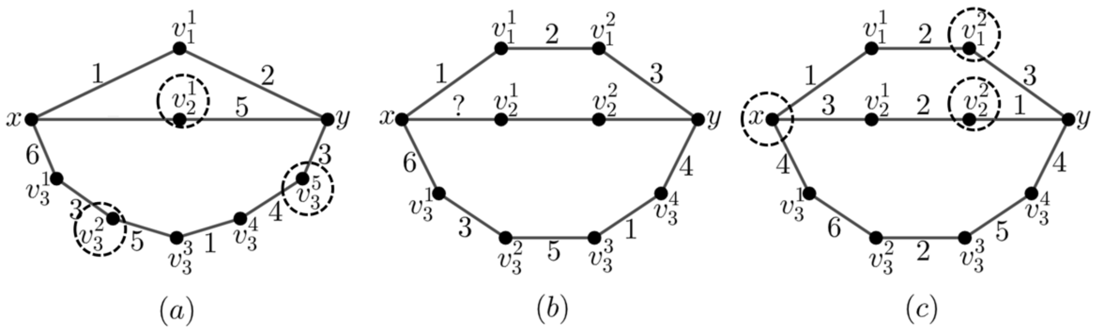

Figure 6. The illustrations of the proof of Case 2 when (**a**) $a_1 = a_2 = 2$ and $a_3 = 6$; (**b**) $a_1 = a_2 = 3$ and $a_3 = 5$; (**c**) $a_1 = a_2 = 3$ and $a_3 = 5$.

To prove the upper bound, we define a strong 3-rainbow coloring of $\theta(a_1, a_1, a_3)$ as given in Figure 7.

Case 3. $a_1 = 2$ and $a_3 = 5$ or $a_3 \geq 7$, or $a_1 = 3$ and $a_3 = 4$ or $a_3 \geq 6$, or $a_1 = 4$ and $a_3 \geq 5$

Let c be a strong 3-rainbow coloring of $\theta(a_1, a_1, a_3)$. It follows by Theorem 1.3 and Observation 2.5 that $srx_3(\theta(a_1, a_1, a_3)) \geq 2a_1 + a_3 - 2$. To prove the upper bound, we define a strong 3-rainbow coloring of $\theta(a_1, a_1, a_3)$ as follows. For $a_1 = 2$ and $a_3 = 5$ or $a_3 \geq 7$, assign color 1 to the edges xv_1^1 and v_2^1y, color 2 to the edges v_1^1y and xv_2^1, and colors $3,4, \dots, a_3 + 2$ to the remaining a_3 edges of $\theta(2,2,a_3)$. For $a_1 = 3$ and $a_3 = 4$ or $a_3 \geq 6$, assign color 1 to the edges xv_1^1 and v_2^2y, color 2 to the edges v_1^2y and xv_2^1, and colors $3,4, \dots, a_3 + 4$ to the remaining $a_3 + 2$ edges of $\theta(3,3,a_3)$. For $a_1 = 4$ and $a_3 \geq 5$, assign color 1 to the edges xv_1^1 and v_2^3y, color 2 to the edges $v_1^2v_1^3$ and $v_2^1v_2^2$, and colors $3,4, \dots, a_3 + 6$ to the remaining $a_3 + 4$ edges of $\theta(4,4,a_3)$.

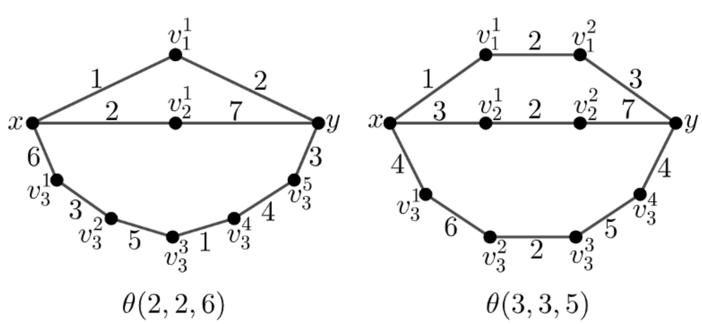

Figure 7. Strong 3-rainbow colorings of $\theta(2,2,6)$ and $\theta(3,3,5)$.

Now, we show that there exists a rainbow Steiner S-tree for every set S of three vertices of $\theta(a_1, a_1, a_3)$. First, consider $a_1 = 2$ and $a_3 = 5$ or $a_3 \geq 7$. Observe that the edge-coloring of $\theta(2,2,a_3)$ above assigns two colors to all edges of C_4 which has the same pattern as an edge-coloring given in Figure 1, and assign $a_3 + 2$ colors to all edges of C_{2+a_3}. It means if $S \subseteq V(C_4)$ or $S \subseteq V(C_{2+a_3})$, then there exists a rainbow Steiner S-tree contained in C_4 or C_{2+a_3}, respectively. Therefore, we assume that $S = \{v_1^1, v_2^1, v_3^p\}$ for $p \in [1, a_3 - 1]$. By the edge-coloring of $\theta(2,2,a_3)$, we also can show that there exists a rainbow Steiner $\{x, v_1^1, v_2^1\}$-tree T_1, a rainbow Steiner $\{v_1^1, v_2^1, y\}$-tree T_2, a rainbow $x - v_3^p$ geodesic T_3, and a rainbow $v_3^p - y$ geodesic T_4 such that $c(E(T_1)) \cap c(E(T_3)) = \emptyset$ and $c(E(T_2)) \cap c(E(T_4)) = \emptyset$. Thus, the tree $T_1 \cup T_3$ or $T_2 \cup T_4$ is a rainbow Steiner S-tree. A similar argument applies for case $a_1 = 3$ and $a_3 = 4$ or $a_3 \geq 6$, or $a_1 = 4$ and $a_3 \geq 5$.

Case 4. $a_1 \geq 5$ and $a_3 \geq 6$

Let c be a strong 3-rainbow coloring of $\theta(a_1, a_1, a_3)$. It follows by Theorem 1.3 and Observation 2.5 that $srx_3(\theta(a_1, a_1, a_3)) \geq 2a_1 + a_3$. Furthermore, since $\|\theta(a_1, a_1, a_3)\| = 2a_1 + a_3$, we have $srx_3(\theta(a_1, a_1, a_3)) = 2a_1 + a_3$ by equation (1). ∎

3. CONCLUSIONS

In this paper, we obtained that $\|H\| - 2$ is the sharp upper bound for $srx_3(H)$ where H is a connected graph containing exactly three edge-disjoint cycles. We also determined the exact values of srx_3 of theta graph $\theta(a_1, a_2, a_3)$ for certain values of a_1, a_2, and a_3. There are many other classes of graphs containing three cycles that have not been studied in this paper. Hence, it is interesting to continue the study by determining the exact values of srx_3 of the graphs. For further study, it is also interesting to study the srx_3 of graphs containing at least four cycles.

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