The Structures of Non-Coprime Graphs for Finite Groups from Dihedral Groups with Regular Composite Orders

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Abstract

For any finite group, the non-coprime graph of the group is a graph with vertices consisting of all non-identity elements of the group. Two different vertices are considered adjacent if their orders are not coprime, meaning their greatest common divisor (gcd) is not equal to one. Misuki provides the structure of the non-coprime graph for the dihedral group when the order is a prime power. We establish a more general property for cases where the order of the group is a regular composite number, discovering that the structure of the non-coprime graph for a dihedral group can be partitioned into some number of complete subgraphs.

Keywords: dihedral group; disjoint subgraph; non-coprime graph.

Abstrak

Untuk sebarang grup hingga, graf non-coprime dari grup tersebut adalah graf dengan simpul yang terdiri dari semua elemen non-identitas dari grup tersebut. Dua simpul yang berbeda dianggap bertetangga jika orde mereka tidak saling prima, artinya pembagi terbesar bersama (gcd) mereka tidak sama dengan satu. Misuki memberikan struktur graf non-coprime untuk grup dihedral ketika ordenya adalah pangkat prima. Pada studi ini didapatkan sifat yang lebih umum ketika orde grup adalah bilangan komposit biasa. Didapatkan juga bahwa struktur graf non-coprime dari grup dihedral dapat dipartisi menjadi beberapa subgraf lengkap.

Kata Kunci: graf non-koprima; grup dihedral; subgraph disjoin.

1. INTRODUCTION

The current focus of research revolves around the graph representation of finite groups. Ma conducted a study related to group representation called coprime graphs [1], then Mansoori 2016 also conducted research related to group representation called non-coprime graphs [2], then new studies emerged related to more specific group representations in graphs. For example, studies have already been conducted on the characterizations and numerical invariants of several graph representations, including their topological indices. These studies include the coprime graph from a dihedral group, an integer group modulo, or a generalized quaternion group, as well as non-coprime graphs from a dihedral group [3]–[10].

In addition to the mentioned studies, research has also explored various other types of graphs, such as prime graphs, intersection graphs, nilpotent graphs, and commuting graphs. These

investigations have sought to understand the properties, characteristics, and numerical invariants associated with these graph structures. For instance, prime graphs examine the relationships between elements in a group based on their prime divisors, while intersection graphs explore the intersections of subgroups within a given group. Commuting graphs, on the other hand, investigate the relationships between group elements that commute with each other. These diverse graph representations provide valuable insights into different aspects of group theory and its applications in mathematics [11]–[19].

2. METHODS

This research begins by conducting a literature study, namely finding and studying reading sources related to the topic. After conducting a literature study, non-coprime graphs were shaped from dihedral groups for several cases. Then from the case that is constructed, we looked for a pattern as a conjecture, and the conjectures are stated as a theorem after we prove it.

3. RESULTS AND DISCUSSIONS

The dihedral group consists of symmetries inherent to a regular n-polygon, and it adheres to the principles outlined by group axioms. The formal definition is given below.

Definition 1. [6] *The dihedral group, denoted by D_{2n} is the set:*

$$D_{2n} = \{e, a, a^2, \dots, a^{n-1}, b, ab, \dots, a^{n-1}b \mid a^n = b^2 = e, a^{-1} = bab\}, n \geq 3.$$

A graph is a mathematical structure widely employed in various fields such as computer science, network modeling, and others. The graph is defined as follows:

Definition 2. A graph is a mathematical structure consisting of a non-empty set of vertices and a set of edges connecting these vertices. A graph is said to be complete if every vertice is connected with an edge.

The non-coprime graph constitutes a novel method for graph representation, and it was introduced by Mansoori [7]. A non-coprime graph involves the order of the group elements as described in the following definition.

Definition 3. [7] *Let G is a finite group, the non-coprime graph of group G denoted by $\overline{\Gamma_G}$ is a graph with vertices consisting of $\bar{G} = G - \{e\}$ and two different vertices $x, y \in G$ is adjacent whenever $(ord(x), ord(y)) \neq 1$. For every $x \in G$, the $ord(x)$ is the smallest positive integer k such that $x^k = e$.*

In this section, we will discuss the shape of non-coprime graphs in dihedral groups. The non-coprime graph for the dihedral group D_6 shown in Table 1. Figure 1 shows the non-coprime graph from the dihedral group D_6. Based on Figure 1, we discovered from the non-coprime graph from the dihedral group D_6 is a combination of two complete graphs $K_2 \cup K_3$.

Table 1. Order from the dihedral group D_6[a]

Component	Order
e	1
a	3
a^2	3
b	2
ab	2
a^2b	2
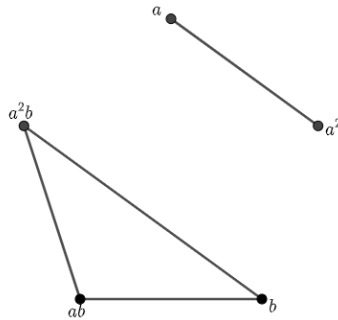
Figure 1. Non-coprime graph from dihedral group D_6

Next, the non-coprime graph from the dihedral group D_8 shown in Table 2. Figure 2 shows a non-coprime graph from the dihedral group D_8. Based on Figure 2, we discovered the shape of the non-coprime graph from the dihedral group D_8 is a complete graph K_7.

Table 2. Order from the dihedral group D_8[a]

Component	Order
e	1
a	4
a^2	2
a^3	4
b	2
ab	2
a^2b	2
a^3b	2

Next, the non-coprime graph from the dihedral group D_{10} shown in Table 3. Figure 3 shows a non-coprime graph from the dihedral group D_{10}. Based on Figure 3, we discovered the shape of the non-coprime graph from the dihedral group D_{10} is the combination of two complete graphs $K_4 \cup K_5$. Using a similar method, we discover the shape of the non-coprime of the dihedral groups $D_{14}, D_{16},$ and D_{18} as well.
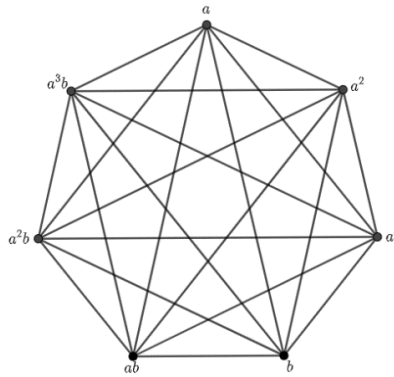
Figure 2. Non-coprime graph from dihedral group D_8

Table 3. Order from the dihedral group D_{10}ᵃ

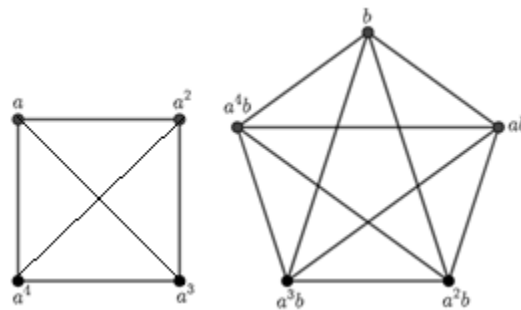| Component	Order
e	1
a	5
a^2	5
a^3	5
a^4	5
b	2
ab	2
a^2b	2
a^3b	2
a^4b	2

Figure 3. Non-coprime graph from dihedral group D_{10}

Figure 4 shows a non-coprime graph from the dihedral group D_{14}. Based on Figure 4, we discovered the shape of the non-coprime graph from the dihedral group D_{14} is a combination of two complete graphs $K_6 \cup K_7$. Figure 5 shows a non-coprime graph from the dihedral group D_{16}. Based on this figure, we discovered the shape of the non-coprime graph from the dihedral group D_{16} is a complete graph K_{15}.
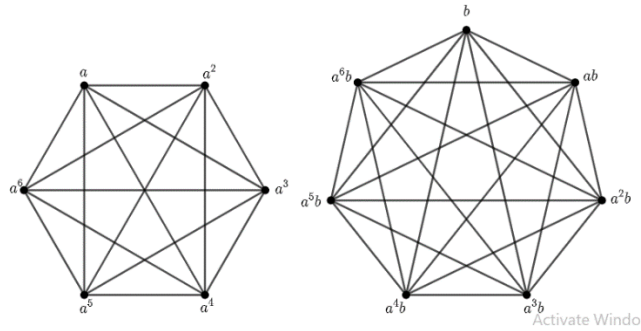
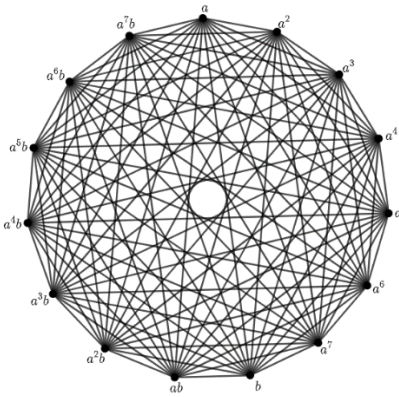**Figure 4**. Non-coprime graph from dihedral group D_{14}

Figure 5. Non-coprime graph from the dihedral group D_{16}

Figure 6 shows a non-coprime graph from the dihedral group D_{18}. Based on Figure 6, we discovered the shape of the non-coprime graph from the dihedral group D_{18} is a combination of two complete graphs $K_8 \cup K_9$. And with several examples of the shapes of non-coprime graphs obtained previously, we obtained several theorems related to the characteristics of non-coprime graphs of dihedral groups, which is more general than what Misuki gave [20].
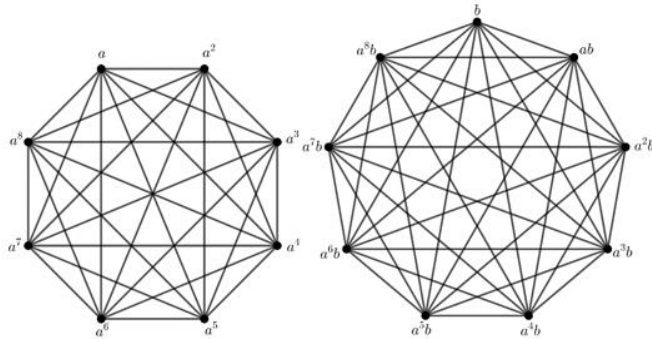

Figure 6. Non-coprime graph from dihedral group D_{18}

Misuki, in his study, already found interesting properties of non-coprime graphs in the following results.

Theorem 1. [4] *Given D_{2n} as the dihedral group. If $n = 2^k$ for some $k \in \mathbb{N}$, then $\overline{\Gamma_{D_{2n}}}$ is a complete graph.*

Apart from the above properties, a non-coprime graph in a dihedral group where n is a prime number has other properties stated in the following theorem.

Theorem 2. [4] *Given D_{2n} as the dihedral group. If $n = p^k$ for some $k \in \mathbb{N}$, and p is an odd prime. Then $\overline{\Gamma_{D_{2n}}}$ consist of two disjoint complete graphs.*

The last property given by Misuki in his study related to non-coprime graphs in the dihedral group where n is a prime number is described in the following theorem.

Theorem 3. [4] *Given H as a non-trivial subgroup of the dihedral group D_{2n}. If $n = p^m$ then the non-coprime graph of H is a trivial graph or a complete graph or consist of two complete graphs.*

Based on the result of the non-coprime graph given by Misuki, we give the more general result of the structure of the non-coprime graph in the dihedral group. Initially, it was observed that it is always possible to identify a subgraph within which forms a complete graph.

Theorem 4. *Given D_{2n} is a dihedral group. If $n = p_1^{k_1} p_2^{k_2} \ldots p_m^{k_m}$, where $p_1, p_2, \ldots, p_m$ are distinct odd primes, then $\overline{\Gamma_{D_{2n}}}$ has a complete subgraph.*

Proof: Let G a subset of $D_{2n} - \{e\}$, where $G = \{b, ab, a^2b, \ldots, a^{n-1}b\}$, because G consists of all reflection elements; consequently, 2 divides each order of $y \in G$, so that each element in G is neighboring, consequently $\overline{\Gamma_{D_{2n}}}$ has a complete subgraph. ∎

Second, we give a more general result of Theorem 2.

Theorem 5. *Given D_{2n} is a dihedral group. If $n = p_1^{k_1} p_2^{k_2} \ldots p_m^{k_m}$, where $p_1, p_2, \ldots, p_m$ are distinct odd prime and $k_1, k_2, \ldots, k_m \in \mathbb{N}$, then $\overline{\Gamma_{D_{2n}}}$ has $m + 1$ disjoint complete subgraph.*

Proof: Define $m + 1$ set, the first set V_1 is a set of elements with order 2, and the second set V_2 is a set of all elements in order o_1, such that $p_1 | o_1$. And for $i = 3, \ldots, m$, the i-set V_i is a set of an element with odd order o_{i-1}, such that $p_{i-1} | o_{i-1}$ and p_j not divide o_{i-1} for all $1 \leq j \leq i - 2$. Noted that, since $2 \nmid n$, all elements in V_j have an odd order for $j > 1$. Let $x \in V_i$ and $y \in V_j$, without loss of generality, let's assume that $1 < i < j$. We have $p_{i-1} | ord(x)$ and $p_l \nmid ord(x)$ for all $l < i - 1$ and also $p_{j-1} | ord(y)$ and $p_r \nmid ord(y)$ for all $r < j - 1$, thus $(ord(x), ord(y)) = 1$, then x and y are not adjacent. So, we can conclude that D_{2n} has $m + 1$ disjoint component.

Now we need to prove that every component is a complete component. Note that V_1 is a complete graph by Theorem 4. And for $i > 1$, every $x, y \in V_i$, we have $p_{i-1} | ord(x)$ and $p_{i-1} | ord(y)$, so x, y are adjacent. Hence V_i is a complete component. ∎

We found a sufficient condition for the non-coprime graph to be connected.

Theorem 6. *Given D_{2n} is a dihedral group. If $2|n$ then $\overline{\Gamma_{D_{2n}}}$ is a connected graph.*

Proof: Let $x, y \in D_{2n} - \{e\}$ arbitrary and $n = p_1^{k_1} p_2^{k_2} \dots p_m^{k_m}$ where $p_1, p_2, \dots, p_m$ are distinct odd prime and $k_1, k_2, \dots, k_m \in \mathbb{N}$. We can assume that $p_i | ord(x)$ and $p_j | ord(y)$ for some $i, j \in \mathbb{N}$, then there exists $a, b \in D_{2n} - \{e\}$ such that $ord(a) = 2p_i$ and $ord(b) = 2p_j$. Note that there is always an element of D_{2n} with order 2, named it c. Then we have a path $x - a - c - b - y$, since x, y arbitrary, then $\overline{\Gamma_{D_{2n}}}$ is a connected graph. ∎

Now we can give the direct result of Theorem 5. If 2 does not divide n, then the non-coprime graph $\overline{\Gamma_{D_{2n}}}$ is not connected.

Corollary 2. *Given D_{2n} is a dihedral group. If $n = p_1^{k_1} p_2^{k_2} \dots p_m^{k_m}$, and $m \geq 2$ where $p_1, p_2, \dots, p_m$ are distinct odd prime and $k_1, k_2, \dots, k_m \in \mathbb{N}$, $\overline{\Gamma_{D_{2n}}}$ is not a connected graph.*

4. CONCLUSIONS

Based on this study, we can conclude that a non-coprime graph of the dihedral group has a structure that consists of some disjoint complete subgraph.

REFERENCES

[1] X. Ma, H. Wei, and L. Yang, "The Coprime graph of a group," *International Journal of Group Theory*, vol. 3, no. 3, pp. 13–23, 2014, doi: 10.22108/ijgt.2014.4363.

[2] F. Mansoori, A. Erfanian, and B. Tolue, "Non-coprime graph of a finite group," *AIP Conf Proc*, vol. 1750, no. June 2016, 2016, doi: 10.1063/1.4954605.

[3] A. G. Syarifudin, Nurhabibah, D. P. Malik, and I. G. A. W. dan Wardhana, "Some characterizatsion of coprime graph of dihedral group D2n," *J Phys Conf Ser*, vol. 1722, no. 1, 2021, doi: 10.1088/1742-6596/1722/1/012051.

[4] N. Nurhabibah, A. G. Syarifudin, and I. G. A. W. Wardhana, "Some Results of The Coprime Graph of a Generalized Quaternion Group Q_4n," *InPrime: Indonesian Journal of Pure and Applied Mathematics*, vol. 3, no. 1, pp. 29–33, 2021, doi: 10.15408/inprime.v3i1.19670.

[5] A. G. Syarifudin, I. G. A. W. Wardhana, N. W. Switrayni, and Q. Aini, "The Clique Numbers and Chromatic Numbers of The Coprime Graph of a Dihedral Group," *IOP Conf Ser Mater Sci Eng*, vol. 1115, no. 1, p. 012083, 2021, doi: 10.1088/1757-899x/1115/1/012083.

[6] A. Gazir S, I. G. A. W. Wardhana, N. W. Switrayni, and Q. Aini, "Some Properties of Coprime Graph of Dihedral Group D_2n When n is a Prime Power," *Journal of Fundamental Mathematics and Applications (JFMA)*, vol. 3, no. 1, pp. 34–38, 2020, doi: 10.14710/jfma.v3i1.7413.

[7] M. Masriani, R. Juliana, A. G. Syarifudin, I. G. A. W. Wardhana, I. Irwansyah, and N. W. Switrayni, "SOME RESULT OF NON-COPRIME GRAPH OF INTEGERS MODULO n GROUP FOR n A PRIME POWER," *Journal of Fundamental Mathematics and Applications (JFMA)*, vol. 3, no. 2, pp. 107–111, 2020, doi: 10.14710/jfma.v3i2.8713.

[8] M. N. Husni, H. Syafitri, A. M. Siboro, A. G. Syarifudin, Q. Aini, and I. G. A. W. Wardhana, "THE HARMONIC INDEX AND THE GUTMAN INDEX OF COPRIME GRAPH OF INTEGER GROUP MODULO WITH ORDER OF PRIME POWER," *BAREKENG: Jurnal Ilmu Matematika dan Terapan*, vol. 16, no. 3, pp. 961–966, Sep. 2022, doi: 10.30598/barekengvol16iss3pp961-966.

[9]   N. I. Alimon, N. H. Sarmin, and A. Erfanian, "The Szeged and Wiener indices for coprime graph of dihedral groups," in *AIP Conference Proceedings*, American Institute of Physics Inc., Oct. 2020. doi: 10.1063/5.0018270.

[10] M. R. Gayatri, Q. Aini, Z. Y. Awanis, S. Salwa, and I. G. A. W. Wardhana, "The Clique Number and The Chromatics Number Of The Coprime Graph for The Generalized Quarternion Group," *JTAM (Jurnal Teori dan Aplikasi Matematika)* , vol. 7, no. 2, pp. 409–416, 2023, doi: 10.31764/jtam.v7i2.13099.

[11] A. v. Zavarnitsine, "Recognition of finite groups by the prime graph," *Algebra and Logic*, vol. 45, no. 4, pp. 220–231, 2006, doi: 10.1007/s10469-006-0020-9.

[12] D. S. Ramdani, I. G. A. W. Wardhana, and Z. Y. Awanis, "THE INTERSECTION GRAPH REPRESENTATION OF A DIHEDRAL GROUP WITH PRIME ORDER AND ITS NUMERICAL INVARIANTS," *BAREKENG: Jurnal Ilmu Matematika dan Terapan*, vol. 16, no. 3, pp. 1013–1020, Sep. 2022, doi: 10.30598/barekengvol16iss3pp1013-1020.

[13] N. Nurhabibah, A. Gazir Syarifudin, I. Gede Adhitya Wisnu Wardhana, and Q. Aini, "Eigen Mathematics Journal The Intersection Graph of a Dihedral Group," *Eigen Mathematics Journal*, vol. 4, no. 2, pp. 68–73, 2021, doi: 10.29303/emj.v4i2.119.

[14] E. Y. Asmarani, A. G. Syarifudin, G. Adhitya, W. Wardhana, and W. Switrayni, "Eigen Mathematics Journal The Power Graph of a Dihedral Group," *Eigen Mathematics Journal*, vol. 4, no. 2, pp. 80–85, 2021, doi: 10.29303/emj.v4i2.117.

[15] N. I. Alimon, N. H. Sarmin, and A. Erfanian, "The topological indices of the non-commuting graph for symmetric groups," *ASM Science Journal*, vol. 13, pp. 1–5, 2020, doi: 10.32802/asmscj.2020.sm26(1.28).

[16] M. Jahandideh, N. H. Sarmin, and S. M. S. Omer, "The topological indices of non-commuting graph of a finite group," *International Journal of Pure and Applied Mathematics*, vol. 105, no. 1, pp. 27–38, 2015, doi: 10.12732/ijpam.v105i1.4.

[17] L. R. W. Putra, Z. Y. Awanis, S. Salwa, Q. Aini, and I. G. A. W. Wardhana, "THE POWER GRAPH REPRESENTATION FOR INTEGER MODULO GROUP WITH POWER PRIME ORDER," *BAREKENG: Jurnal Ilmu Matematika dan Terapan*, vol. 17, no. 3, pp. 1393–1400, Sep. 2023, doi: 10.30598/barekengvol17iss3pp1393-1400.

[18] D. P. Malik *et al.*, "Graf Nilpoten dari Gelanggang Bilangan Bulat Modulo Berorde Pangkat Prima (A Note on Nilpotent Graph of Ring Integer Modulo with Order Prime Power)," *JMPM: Jurnal Matematika dan Pendidikan Matematika*, vol. 8, no. 1, pp. 28–33, 2023, doi: 10.26594/jmpm.v8i1.2920.

[19] E. Y. Asmarani, S. T. Lestari, D. Purnamasari, A. G. Syarifudin, S. Salwa, and I. G. A. W. Wardhana, "The First Zagreb Index, The Wiener Index, and The Gutman Index of The Power of Dihedral Group," *CAUCHY: Jurnal Matematika Murni dan Aplikasi*, vol. 7, no. 4, pp. 513–520, May 2023, doi: 10.18860/ca.v7i4.16991.

[20] W. U. Misuki, I. G. A. W. Wardhana, N. W. Switrayni, and Irwansyah, "Some results of non-coprime graph of the dihedral group D2n for n a prime power," *AIP Conf Proc*, vol. 2329, no. February, 2021, doi: 10.1063/5.0042587.