

## Some Characteristics of the Prime Graph of Integer Modulo Groups

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### Abstract

The notion of the prime graph of a ring  $R$  was first introduced by Bhavanari, Kuncham, and Dasari in 2010. The prime graph of a ring  $R$ , denoted by  $PG(R)$  is a graph whose vertices are all elements of the ring, where two distinct vertices  $x$  and  $y$  are adjacent if and only if  $xRy = 0$  or  $yRx = 0$ . In this paper, we study the forms and properties of the prime graph of integer modulo group, and some examples of the number of its spanning trees. In this paper, it is found that for all  $n$ , the maximum degree of vertices of  $PG(\mathbb{Z}_n)$  is  $n - 1$  and the minimum degree of its vertices is 1. Then, we show that for all  $n$ ,  $PG(\mathbb{Z}_n)$  is neither a Hamiltonian graph nor an Eulerian graph. We also found some examples of the number of its spanning trees.

**Keywords:** prime graph; spanning trees; Hamiltonian graph.

### Abstrak

Konsep mengenai graf prima dari suatu gelanggang  $R$  pertama kali diperkenalkan oleh Bhavanari, Kuncham, dan Dasari pada tahun 2010. Graf prima dari suatu gelanggang  $R$ , yang dinotasikan dengan  $PG(R)$ , adalah suatu graf yang simpul-simpulnya merupakan semua elemen dari gelanggang tersebut dengan dua buah simpul  $x$  dan  $y$  yang berbeda akan bertetangga jika dan hanya jika  $xRY = 0$  atau  $YRx = 0$ . Di dalam penelitian ini, dikaji mengenai bentuk-bentuk dan sifat-sifat dari  $PG(\mathbb{Z}_n)$ , dan beberapa contoh dari banyak pohon pembangunnya. Pada penelitian ini, ditemukan hasil bahwa untuk setiap  $n$ , derajat maksimal dari simpul-simpul di  $PG(\mathbb{Z}_n)$  adalah  $n - 1$  dan derajat minimum dari simpul-simpulnya adalah 1. Hasil selanjutnya yaitu, untuk setiap  $n$ ,  $PG(\mathbb{Z}_n)$  bukan merupakan suatu graf Hamiltonian atau graf Eulerian. Ditemukan juga beberapa contoh dari banyaknya pohon pembangun dari  $PG(\mathbb{Z}_n)$ .

**Kata Kunci:** graf prima; pohon pembangun; graf Hamiltonian.

**2020MSC:** 05E16, 05C90, 20C05

## 1. INTRODUCTION

We can define a graph on an algebraic structure like groups or rings. In 2021, Syarifudin et. al studied some characteristics of the coprime graph of a dihedral group [1]. Meanwhile, in 2020, Fatahillah and Switrayni [2] researched the characteristics of zero divisor graphs of a certain polynomial ring. In graph theory, there is a term similar to a zero divisor graph called the *prime graph of a ring*. The term *prime graph of a ring* was first introduced by Bhavanari, Kuncham, and Dasari in 2010 [3]. Several studies define graphs on a group structure, like the coprime graph, the non-coprime graph,

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or that represent an integer modulo group, a dihedral group, a generalized quaternion group, the intersection graph, the prime graph, the topological indices of the coprime graph [4]–[15]. In the algebraic structure ring, there is also some study of a graph that defines the ring, like that zero divisor graph or prime graph [15]–[17].

In graph theory, there is a topic called the number of spanning trees on a graph. In [18], a tree is defined as a connected graph that has no cycle. The number of spanning trees of a connected graph is defined as the number of trees that can be constructed from the graph which passes through all of its vertices. In [4], the authors obtained some formulas about the number of spanning trees of the fan graph, complete graph, and wheel graph. Then in 2019, Maulana and Switrayni [18] obtained some formulas about the number of spanning trees of the barbell graph. In this article, we will give some characteristics of the prime graph of the integer modulo group.

## 2. METHODS

The method used in this paper is a literature study, by reviewing the definition of a prime graph, Hamiltonian graph, and Eulerian graph and several theorems about graphs and its spanning trees. Moreover, we also form the prime graph of  $\mathbb{Z}_n$  for some  $n$ , followed by analyzing its properties and counting its spanning trees. The properties of  $PG(\mathbb{Z}_n)$  that are analyzed in this paper are its maximum and minimum degree, diameter, edge connectivity, vertex connectivity, the minimum number of its triangles, and the type of graph that is formed. In addition, we compute the number of spanning trees of  $PG(\mathbb{Z}_n)$  for  $n = 2p$ ,  $n = 3p$ , and  $n = 5p$  for some prime number  $p$ , using Maple 13, followed by making several conjectures about  $PG(\mathbb{Z}_n)$  and proving them.

## 3. RESULTS AND DISCUSSIONS

One of the main focuses of this paper is to obtain some characteristics of  $PG(\mathbb{Z}_n)$ , hence we need to know the definition of the prime graph of a ring. The definition of the prime graph of a ring is given by the definition below.

**Definition 1.** [11] Let  $R$  be a ring. A graph  $G(V, E)$  is said to be a prime graph of  $R$  (denoted by  $PG(R)$ ) if  $V = R$  and  $E = \{\overline{xy} | xRy = 0 \text{ or } yRx = 0 \text{ and } x \neq y\}$ .

**Example 1.** Consider  $\mathbb{Z}_6$ , the ring of integers modulo 6. Then,  $PG(\mathbb{Z}_6)$  is shown in Figure 1.

According to [10], we can construct a tree from a connected graph  $G$  by eliminating some edges so that there is no cycle in graph  $G$ . In this paper, we use Maple 13 to determine the number of spanning trees of  $PG(\mathbb{Z}_n)$ . Some basic graph terminology throughout this article is given by the next definitions.

**Definition 2.** [19] Let  $G(V, E)$  be a graph with vertices  $V$  and edges  $E$ . The maximum degree of graph  $G$  is the largest number of edges that are incident to any single vertex in the graph. And the minimum degree of graph  $G$  is the smallest number of edges that are incident to any single vertex in the graph.

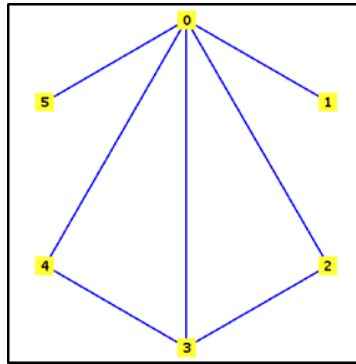


Figure 1. Prime graph of a ring  $\mathbb{Z}_6$  ( $PG(\mathbb{Z}_6)$ ).

**Definition 3.** [20] Let  $G(V, E)$  be a connected graph with vertices  $V$  and edges  $E$ . The diameter graph  $G$  is the longest shortest path between any two vertices in the graph.

**Definition 4.** [21] Let  $G(V, E)$  be a connected graph with vertices  $V$  and edges  $E$ . The edge connectivity of a graph  $G$  is defined as the minimum number of edges that need to be removed to disconnect the graph. And the vertex connectivity of graph  $G$  is defined as the minimum number of vertices that need to be removed to disconnect the graph.

### 3.1 Some Characteristics of $PG(\mathbb{Z}_n)$

In this section, we will discuss some characteristics of  $PG(\mathbb{Z}_n)$ . For example the maximum and minimum degree of vertices in  $PG(\mathbb{Z}_n)$ , the diameter of  $PG(\mathbb{Z}_n)$ , edge connectivity, and vertex connectivity. The following theorem can help us to determine the degree of vertices in  $PG(\mathbb{Z}_n)$  and the diameter of  $PG(\mathbb{Z}_n)$ .

**Theorem 1.** Let  $PG(\mathbb{Z}_n)$  be a prime graph of a ring  $\mathbb{Z}_n$ . Then, for all  $n$  the following conditions hold:

- (i) The maximum degree of vertices in  $PG(\mathbb{Z}_n)$  is  $n - 1$  and the minimum degree of vertices in  $PG(\mathbb{Z}_n)$  is 1.
- (ii) The diameter of  $PG(\mathbb{Z}_n)$  is 2.

*Proof.* (i) Let  $PG(\mathbb{Z}_n)$  is a prime graph of a ring  $\mathbb{Z}_n$ . If the maximum degree of vertices in  $PG(\mathbb{Z}_n)$  is greater than  $n - 1$ , then there are multiple edges. It contradicts the fact that  $PG(\mathbb{Z}_n)$  is a simple graph (no multiple edges). The vertex with the maximum degree is vertex 0 because for each  $a \in \mathbb{Z}_n - \{0\}$ ,  $a \cdot 0 = 0$ . The vertex with degree 1 is vertex 1 because vertex 1 is only adjacent to vertex 0. Hence its degree is 1.

(ii) In  $PG(\mathbb{Z}_n)$ , vertex 0 has the maximum degree  $n - 1$ . Since vertex 0 is adjacent to all non-zero vertices of  $PG(\mathbb{Z}_n)$ , then the shortest path that connects any two non-zero vertices in  $PG(\mathbb{Z}_n)$  must pass through vertex 0. Hence  $d(u, v) = 2$ , for any non-zero vertex  $u$  and  $v \in V(PG(\mathbb{Z}_n))$  ■

As a consequence, we have the following result.

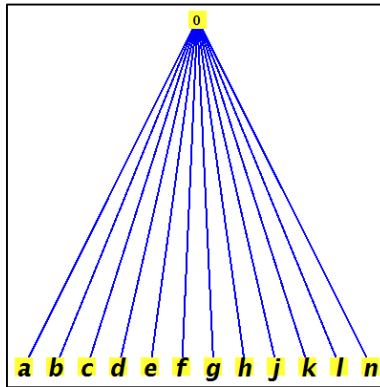
**Corollary 1.** Let  $PG(\mathbb{Z}_n)$  be a prime graph of a ring  $\mathbb{Z}_n$ . Then edge-connectivity and vertex-connectivity of  $PG(\mathbb{Z}_n)$  are equal to one.

*Proof.* By (i) in Theorem 1, we know that for all  $n$ ,  $0$  has the maximum degree  $n - 1$ . Therefore  $0$  is adjacent to all non-zero vertex of  $PG(\mathbb{Z}_n)$ . Noted that  $1$  is only adjacent to  $0$ , then by deleting the vertex  $0$ ,  $PG(\mathbb{Z}_n)$  will be a disconnected graph. So, the edge-connectivity of  $PG(\mathbb{Z}_n)$  is 1. And vertex  $1$  has degree 1 (the minimum degree) since  $1$  is only adjacent to  $0$ . Therefore, by deleting the edge that connects vertex  $1$  and  $0$ ,  $PG(\mathbb{Z}_n)$  will be a disconnected graph, hence the vertex-connectivity of  $PG(\mathbb{Z}_n)$  is 1. ■

According to [10], a cycle of length 3 is called a triangle. The following theorem lets us determine the minimum number of triangles of  $PG(\mathbb{Z}_n)$ .

**Corollary 2.** Let  $Z(\mathbb{Z}_n)$  be a set of all unordered pairs  $(a, b)$ , with  $a \neq b, a \neq 0$  and  $b \neq 0, a, b \in \mathbb{Z}_n$  such that  $a \cdot b = 0$ . Then, the minimum number of triangles of  $PG(\mathbb{Z}_n)$  is  $|Z(\mathbb{Z}_n)|$ .

*Proof.* Let  $PG(\mathbb{Z}_n)$  be an arbitrary prime graph of  $\mathbb{Z}_n$ . If  $n$  is prime, then  $PG(\mathbb{Z}_n)$  contains no cycle, hence it is a tree. Here is the illustration of  $PG(\mathbb{Z}_n)$  if  $n$  is prime.



**Figure 2.**  $PG(\mathbb{Z}_n)$  if  $n$  is prime formed a star graph.

Based on Figure 2, if  $n$  is composite then  $|Z(\mathbb{Z}_n)| \neq 0$ . Therefore, there is an edge that connects two non-zero vertices of  $PG(\mathbb{Z}_n)$ . Without loss of generality, let  $|Z(\mathbb{Z}_n)| = 1$  with  $a \cdot b = 0$ . Then, there is an addition of edge in  $PG(\mathbb{Z}_n)$  which connects  $a$  and  $b$ . By adding one edge that connects  $a$  and  $b$ , we have a triangle  $0 \rightarrow a \rightarrow b \rightarrow 0$ . If  $|Z(\mathbb{Z}_n)|$  is increasing, then the number of triangles of  $PG(\mathbb{Z}_n)$  is also increasing. If the number of triangles of  $PG(\mathbb{Z}_n)$  is less than  $|Z(\mathbb{Z}_n)|$ , then there is  $(a, b) \in Z(\mathbb{Z}_n)$  but there is no edge connecting  $a$  and  $b$ . It contradicts the definition of a prime graph. So, there must be an edge connecting  $a$  and  $b$ . ■

In graph theory, there is a term called a Hamiltonian graph. Graph  $G$  is called a Hamiltonian if graph  $G$  contains a Hamiltonian cycle [10]. According to [10], a cycle is called Hamiltonian if it passes every vertex of  $G$  exactly once. The following theorem tells us that  $PG(\mathbb{Z}_n)$  is not a Hamiltonian graph for any  $n$ .

**Corollary 3.**  $PG(\mathbb{Z}_n)$  is not Hamiltonian for any  $n$ .

*Proof.* Let  $PG(\mathbb{Z}_n)$  be a prime graph with an arbitrary  $n$ . Based on part (i) of Theorem 1, we know that vertex 1 has a degree of 1 for any  $n$ . Therefore, vertex 1 is not contained in any cycle in  $PG(\mathbb{Z}_n)$ . Hence, there is no Hamiltonian cycle in  $PG(\mathbb{Z}_n)$ . Therefore,  $PG(\mathbb{Z}_n)$  is not Hamiltonian. ■

Similar to the concept of the Hamiltonian graph, in [10] graph  $G$  is called Eulerian if it contains an Eulerian trail. A closed trail is called Eulerian if it contains every edge of  $PG(\mathbb{Z}_n)$ . The following theorem gives us the fact that  $PG(\mathbb{Z}_n)$  is not Eulerian for any  $n$ .

**Corollary 4.**  $PG(\mathbb{Z}_n)$  is not Eulerian for any  $n$ .

*Proof.* The proof of this theorem is similar to the proof of Corollary 3. Since vertex 1 has a degree of 1 for any  $n$ , then there is no closed trail in  $PG(\mathbb{Z}_n)$ . Therefore  $PG(\mathbb{Z}_n)$  is not Eulerian for any  $n$ . ■

The vertex set of a graph  $G$  may be split into some disjoint sets. If  $V(G)$  can be partitioned into  $k$  disjoint subsets  $V_1, V_2, \dots, V_k$  (called partite sets), such that every edge of  $G$  joins vertices in two different partite sets, then  $G$  is called a  $k$ -partite graph. Here is a theorem that can help us to determine the maximum partite sets of  $PG(\mathbb{Z}_n)$ .

**Theorem 2.** Let  $PG(\mathbb{Z}_n)$  be a prime graph of a ring  $\mathbb{Z}_n$ . Then,  $PG(\mathbb{Z}_n)$  is a bipartite graph when  $n$  is prime, otherwise, the graph is  $(n - \phi(n))$ -partite.

*Proof.* Let  $PG(\mathbb{Z}_n)$  be a prime graph of a ring  $\mathbb{Z}_n$ . If  $n$  is prime, we can construct two partite sets  $V_0$  and  $V_1$  with  $V_0 = \{0\}$  and  $V_1$  containing all non-zero elements of  $\mathbb{Z}_n$ . So  $PG(\mathbb{Z}_n)$  is a 2-partite graph (bipartite graph). If  $\mathbb{Z}_n$  contains zero divisors, then we can construct a new set  $C = \{a_i \in \mathbb{Z}_n | a_i \text{ is zero divisor}\}$  and the partite sets of  $PG(\mathbb{Z}_n)$  as  $V_{a_i} = \{b \in C | a_i \cdot b \neq 0 \text{ or } b = a_i\}$ . Vertices that are relatively prime with  $n$  can be put in any of those partite sets besides  $V_0 = \{0\}$ . The number of relatively prime elements in  $\mathbb{Z}_n$  is denoted by  $\phi(n)$ . Furthermore, the number of zero-divisors in  $\mathbb{Z}_n$  is equivalent to  $(n - 1) - \phi(n)$ . If  $\mathbb{Z}_n$  contains zero divisors, then the number of partite sets besides  $V_0 = \{0\}$  is  $(n - 1) - \phi(n)$ . Therefore, the maximum number of partite sets of  $PG(\mathbb{Z}_n)$  is  $(n - 1) - \phi(n) + 1 = n - \phi(n)$ . ■

### 3.2 Example of $PG(\mathbb{Z}_n)$ and its Spanning Trees

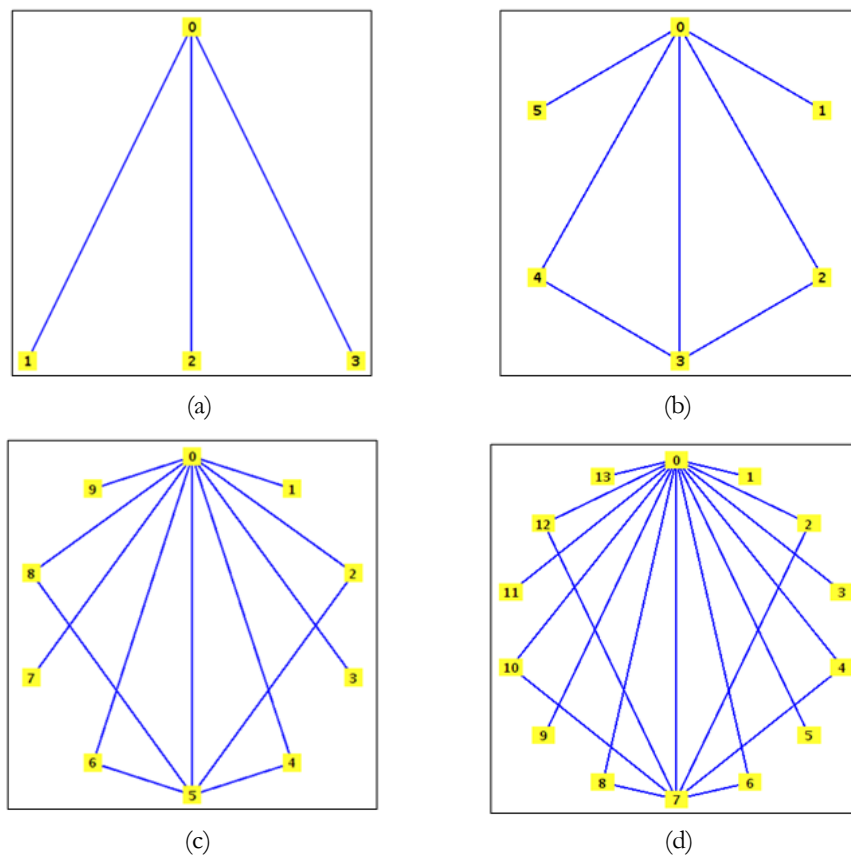
In this section, we will show you some examples of the graph  $PG(\mathbb{Z}_n)$  and their spanning trees. And we will show you there seems no pattern we can formulate to generalize the number of spanning trees of those graphs.

In the beginning, we show you the spanning graph of the group, whenever  $n = 2p$  for some prime number  $p$ . Table 1 consists of several spanning trees of  $PG(\mathbb{Z}_n)$  with  $n = 2 \cdot p$ . To illustrate the information of the spanning tree from Table 1 we can draw the prime graph of those groups in Figure 3.

Figure 3 shows us that in the spanning tree of those graphs, the orders are the multiplication between 2 and a prime is not formed a pattern. Now, we will show you whenever  $n = 3p$  for some prime number  $p$ , and whenever  $n = 5p$ , for some prime number  $p$ .

**Table 1.** Several spanning trees of  $PG(\mathbb{Z}_n)$  with  $n = 2 \cdot p$ .

Prime Graph	Number of Spanning Trees
$PG(\mathbb{Z}_4)$	1
$PG(\mathbb{Z}_6)$	8
$PG(\mathbb{Z}_{10})$	48
$PG(\mathbb{Z}_{14})$	256



**Figure 3.** Prime graph of (a)  $\mathbb{Z}_4$ , (b)  $\mathbb{Z}_6$ , (c)  $\mathbb{Z}_{10}$  and (d)  $\mathbb{Z}_{14}$ .

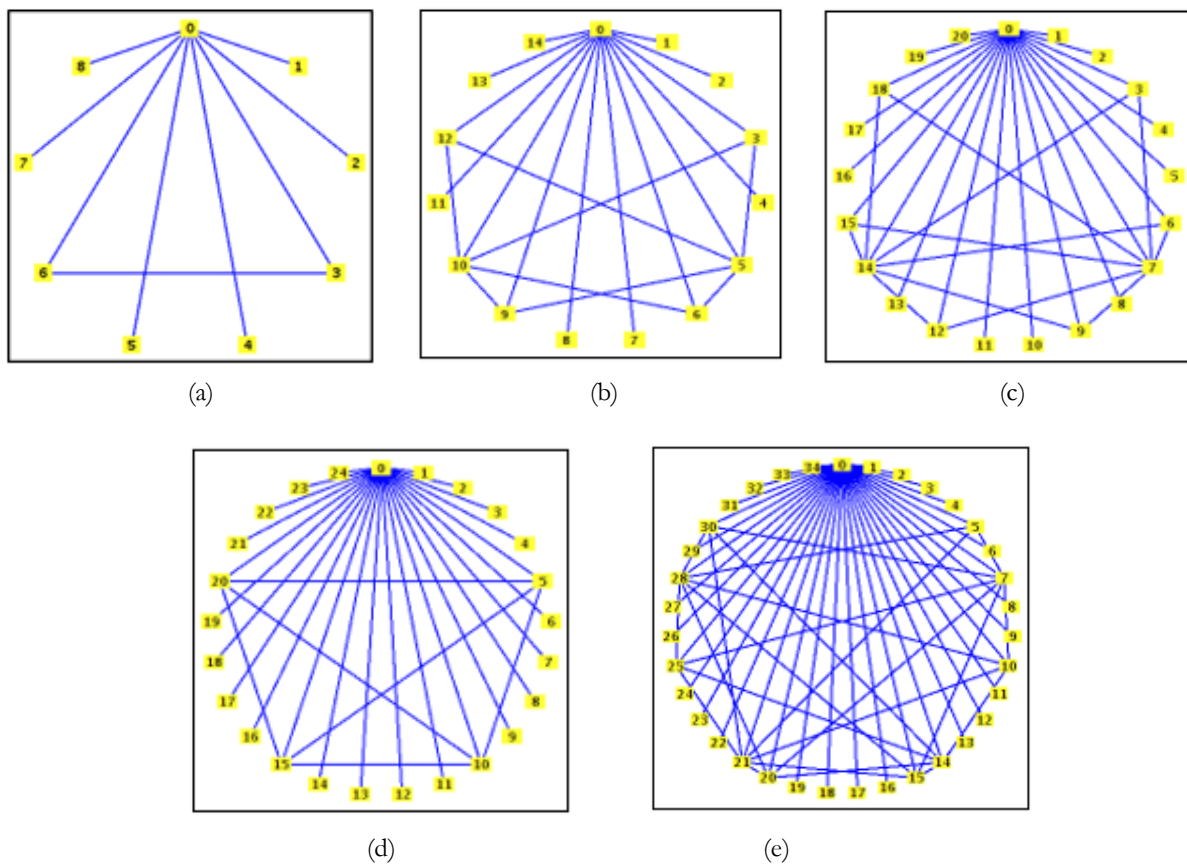
Table 2 consists of several spanning trees of  $PG(\mathbb{Z}_n)$  with  $n = 3 \cdot p$  and Table 3 consists of several spanning trees of  $PG(\mathbb{Z}_n)$  with  $n = 5 \cdot p$ . To illustrate the information of the spanning tree from Table 2 and Table 3, we can draw the prime graph of those groups in Figure 4. Figure 4 confirmed that in the spanning tree of those graphs, the orders are the multiplication between two primes, and it seems comes not to form a pattern.

**Table 2.** Several Spanning Trees of  $PG(\mathbb{Z}_n)$  with  $n = 3 \cdot p$ .

Prime Graph	Number of Spanning Trees
$PG(\mathbb{Z}_9)$	3
$PG(\mathbb{Z}_{15})$	945
$PG(\mathbb{Z}_{21})$	15.309

**Table 3.** Several Spanning Trees of  $PG(\mathbb{Z}_n)$  with  $n = 5 \cdot p$

Prime Graph	Number of Spanning Trees
$PG(\mathbb{Z}_{25})$	125
$PG(\mathbb{Z}_{35})$	8.115.625



**Figure 4.** Prime graph of (a)  $\mathbb{Z}_9$ , (b)  $\mathbb{Z}_{15}$ , (c)  $\mathbb{Z}_{21}$ , (d)  $\mathbb{Z}_{25}$ , and (e)  $\mathbb{Z}_{35}$ .

#### 4. CONCLUSIONS

We found that the prime graph  $PG(\mathbb{Z}_n)$  is bipartite whenever  $n$  is a prime, otherwise, the graph is a  $(n - \phi(n))$ -partite. The prime graph  $PG(\mathbb{Z}_n)$  has the edge-connectivity and the vertex-connectivity equal to 1. In addition, the maximum degree and the minimum degree of vertices in  $PG(\mathbb{Z}_n)$  are  $n - 1$  and 1 respectively. A general formula for counting spanning trees of  $PG(\mathbb{Z}_n)$  is still open for future research.

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