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On Codes Over $\mathcal{R}$ and its Bounds of Some kind of Block Repetition Codes in $\mathcal{R}$

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Abstract

This correspondence determines the lower and upper bounds of the covering radius in some kind of block repetition codes over the finite ring $\mathcal{R} = \mathbb{Z}_2\mathbb{Z}_*$, where $\mathbb{Z}_* = \mathbb{Z}_2 + v\mathbb{Z}_2 + v^2\mathbb{Z}_2$, $v^3 = v$. For covering radii of binary and octonary block repetition code over $\mathcal{R}$ is also discussed. This leads to the convenient formulation of code and arrives at the bounds.

Keywords: block repetition codes; covering radius; different weight; finite ring.

Abstrak

Korespondensi ini menentukan batas bawah dan batas atas dari jari-jari penutup suatu kode blok perulangan pada gelanggang hingga $\mathcal{R} = \mathbb{Z}_2\mathbb{Z}_*$, *dengan* $\mathbb{Z}_* = \mathbb{Z}_2 + v\mathbb{Z}_2 + v^2\mathbb{Z}_2$, $v^3 = v$. *Dibahas juga jari-jari kode blok perulangan biner dan oktonari atas* $\mathcal{R}$. *Diperoleh rumus untuk kode dan batasnya.*

Kata Kunci: kode blok perulangan; penutup jari-jari; berat yang berbeda; gelanggang hingga.

2020MSC: 11T71, 94B05, 11H71.

1. INTRODUCTION

Recently, there has been substantial interest in the class of additive codes. In [1] [2], Delsarte contributes to the algebraic theory of association scheme where the main idea is to characterize the subgroups of the underlying abelian group in a given association scheme.

Additive codes over $\mathbb{Z}_2\mathbb{Z}_4$ have been extensively studied in [3] [4] [5] [6]. In [7] [8], the author gave lower and upper bounds on the covering radius of codes over the finite rings with respect to different distance and they explained the covering radius of various repetition codes.

The above results motivate us to work in this area. In this correspondence, obtain the block repetition codes over $\mathcal{R}$, with respect to different weights such as Lee, Euclidean, Chinese Euclidean, and Homogeneous. At this juncture, the meaning of constructing new codes is to concatenate binary and octonary. These results in the block repetition codes over $\mathcal{R}$, which contain the corresponding $\mathbb{Z}_2$ and $\mathbb{Z}_*$ codes as a subclass.

2. PRELIMINARIES

Let $\mathcal{R}$ be a finite ring, where $\mathcal{R} = \mathbb{Z}_2\mathbb{Z}_*$ and $\mathbb{Z}_* = \mathbb{Z}_2 + v\mathbb{Z}_2 + v^2\mathbb{Z}_2$, $v^3 = v$ and $\mathbb{Z}_2 = \{0,1\}$ is an integer modulo 2. That is, the finite rings $\mathbb{Z}_* = \{0, 1, v, 1+v, v^2, 1+v^2, v+v^2, 1+v+v^2\}$ and $\mathcal{R} =$

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$\{00, 01, 0v, 0a, 0v^2, 0b, 0c, 0d, 10, 11, 1v, 1a, 1v^2, 1b, 1c, 1d\}$, where $a = 1 + v, b = 1 + v^2, c = v + v^2, d = 1 + v + v^2$.

In this section, some preliminary results are given based on [4] and [6]. A non-empty set C is an $\mathcal{R}$-additive code if it is a subgroup of $\mathbb{Z}_2^\alpha \times \mathbb{Z}_*^\beta$. In this case, C is also isomorphic to an abelian structure $\mathbb{Z}_2^\alpha \times \mathbb{Z}_*^\beta$, for some α and β. That C is of type $2^\lambda 8^\mu$ as a group. It follows that it has $|C| = 2^{\alpha+3\beta}$ codewords and the number of order two codewords in C is $|C| = 2^{\alpha+\beta}$.

Let $\phi: \mathbb{Z}_* \to \mathbb{Z}_2^4$, be a Gray map is defined by [9] [10]:

$$\begin{gathered}
\phi(0) = (0, 0, 0, 0),\\
\phi(1) = (0, 1, 0, 1),\\
\phi(v) = (0, 0, 1, 1),\\
\phi(a) = (0, 1, 1, 0),\\
\phi(v^2) = (1, 1, 1, 1),\\
\phi(b) = (1, 0, 1, 0),\\
\phi(c) = (1, 1, 0, 0),\\
\phi(d) = (1, 0, 0, 1).
\end{gathered}$$

In general, the extension Gray map is

$$\rho = \mathbb{Z}_2^\alpha \times \mathbb{Z}_*^\beta \to \mathbb{Z}_2^n, \text{ with } n = \alpha + 3\beta,$$

given by

$$\rho(u, w) = \left(u, \phi(w_1), \cdots \phi(w_\beta)\right), \forall u \in \mathbb{Z}_2^\alpha, \forall \left(w_1, \cdots, w_\beta\right) \in \mathbb{Z}_*^\beta .$$

Then the binary image of a $\mathcal{R}$-additive code under the extended Gray map is called a $\mathcal{R}$-*linear code* of length $n = \alpha + 3\beta$.

The Hamming weight of w, denoted by $wt_H(u)$ and $wt_L(w), wt_E(w), wt_{CE}(w), wt_{Hom}(w)$ the Lee, Euclidean, Chinese Euclidean and Homogeneous weights of w respectively where $u \in \mathbb{Z}_2^\alpha$ and $w \in \mathbb{Z}_*^\beta$.

In [11] [12] are defined as the vector of $y = (y_1, y_2, \cdots, y_n) \in \mathbb{Z}_*$ in table 1.

Table 1. Define for vector $y = (y_1, y_2, \cdots, y_n) \in \mathbb{Z}_*$.

Code	$y \in \mathbb{Z}_*$
$w_L(y)$	0 if $y = 0$; 1 if $y = \{1, a, b, c\}$; 2 if $y = \{v, v^2\}$; and 4 otherwise
$w_E(y)$	0 if $y = 0$; 1 if $y = \{1, d\}$; 4 if $= \{v, v^2, c\}$; and 9 otherwise
$w_{CE}(y)$	0 if $y = 0$; 1 if $y = \{1, d\}$; 2 if $y = \{v, c\}$; 3 if $= \{a, b\}$ and 4 otherwise
$w_{Hom}(y)$	0 if $y = 0$; 2 if $y \neq v^2$; and 4 otherwise

If $c_1, c_2 \in C$ be any two distinct codewords of distance d_D are defined as $d_D(C) = \{d_D(c_1, c_2) | c_1 - c_2 \neq 0 \text{ } and \text{ } c_1, c_2 \in C\}$. The minimum weight of C is $d_D(C) = min\{d_D(c_1, c_2) | c_1 - c_2 \neq 0 \text{ } and \text{ } c_1, c_2 \in C\}$. In C is a *linear code*, then the $d_D(C) = min\{w_D(c) | c \neq 0 \in C\}$. Therefore, $d_D(c_1, c_2) = w_D(c_1 - c_2)$.

Let $C \subseteq \mathcal{R}^n$ is a linear code, where n is the length of code, the number of codewords N and the minimum distance d_D is said to be an (n, N, d_D)-*code* in $\mathcal{R}$. The weights of x are defined as $wt_D(x) = wt_H(u) + wt_D(w)$ with $x = (u, w) \in \mathbb{Z}_2^\alpha \times \mathbb{Z}_*^\beta$, and $u = (u_1, \cdots, u_\alpha) \in \mathbb{Z}_2^\alpha$, $w = \left(w_1, \cdots, w_\beta\right) \in \mathbb{Z}_*^\beta$, where $D = \{$Lee(L), Euclidean(E), Chinese Euclidean(CE) and Homogeneous(Hom)$\}$.

The Gray map defined above is an isometry that transforms the (weights) distances defined over $\mathbb{Z}_2^{\alpha} \times \mathbb{Z}_*^{\beta}$ to the Hamming distance defined over $\mathbb{Z}_2^n$, with $n = \alpha + 3\beta$.

Example 2.1. Let $y = (1\ v\ v^2) \in \mathcal{R}$. Then $wt_H(y) = 3$, $wt_E(y) = 9$, $wt_{CE}(y) = 7$ and $wt_{Hom}(y) = 8$.

3. THE COVERING RADIUS OF CODES AND BLOCK REPETITION CODES IN $\mathcal{R}$

Let C be a code of length n with minimum distance d over a code alphabet $\mathcal{R}$. Then the spheres of radius $\left\lfloor \frac{d-1}{2} \right\rfloor$ around the codewords may not cover the whole space. The least non-negative integer a such that the sphere of radius r around the codewords cover the whole space $\mathcal{R}^n$ is called the *covering radius* of the code. That is, the covering radius of C is

$$\mathbb{R}(C) = \max_{t \in \mathcal{R}^n} \left\{ \min_{c \in C} \{d(t, c)\} \right\}.$$

For a binary code C, its covering radius $r(C)$ is defined as follows

$$r(C) = \max_{t \in \mathbb{F}_2} \left\{ \min_{c \in C} \{d_H(t, c)\} \right\}.$$

The extension of this definition to codes over $\mathcal{R}$ is that the covering radius of a code C is the smallest number a such that the spheres of radius r around the codewords cover $\mathbb{Z}_2^{\alpha} \times \mathbb{Z}_*^{\beta}$. Hence, the covering radius of a code C over $\mathcal{R}$, with respect to the distance(D), is given by

$$r_D(C) = \max_{t \in \mathbb{Z}_2^{\alpha} \times \mathbb{Z}_*^{\beta}} \left\{ \min_{c \in C} \{d_D(t, c)\} \right\},$$

respectively. It is easy to see that $r_D(C)$ is the minimum value r_D such that

$$\mathbb{Z}_2^{\alpha} \times \mathbb{Z}_*^{\beta} = \cup_{c \in C} S_{rD}(c),$$

respectively, where

$$S_{rD}(u) = \left\{ w \in \mathbb{Z}_2^{\alpha} \times \mathbb{Z}_*^{\beta};\ d_D(u, w) \le r_D \right\}, \text{ for } u \in \mathbb{Z}_2^{\alpha} \times \mathbb{Z}_2^{\beta}.$$

In order to determine the covering radius of some classes of block codes over $\mathcal{R}$ are defined. The approach in [1] is used to obtain the covering radius.

Let C^n be a block repetition code over $\mathcal{R}$ is an $\mathcal{R}$-additive code of length $n = \sum_{j=1}^{15} n_j$ with generator matrix

$$G = (\overbrace{01 \cdots 01}^{n_1} \overbrace{0v \cdots 0v}^{n_2} \overbrace{0a \cdots 0a}^{n_3} \overbrace{0v^2 \cdots 0v^2}^{n_4} \overbrace{0b \cdots 0b}^{n_5} \overbrace{0c \cdots 0c}^{n_6} \overbrace{0d \cdots 0d}^{n_7} \overbrace{10 \cdots 10}^{n_8} \overbrace{11 \cdots 11}^{n_9} \overbrace{1v \cdots 1v}^{n_{10}}$$
$$\overbrace{1a \cdots 1a}^{n_{11}} \overbrace{1v^2 \cdots 1v^2}^{n_{12}} \overbrace{1b \cdots 1b}^{n_{13}} \overbrace{1c \cdots 1c}^{n_{14}} \overbrace{1d \cdots 1d}^{n_{15}}),$$ where $a = 1 + v, b = 1 + v^2, c = v + v^2, d = 1 + v + v^2$.

If, for a fixed $1 \leq i \leq 15$. For all $1 \leq j \neq i \leq 15$, $n_j = 0$, then the code $C^n = C^{n_i}$ is denoted by C_i. Therefore, the fifteen basic codes are given in Table 2.

Table 2. The fifteen basic codes

Generator Matrix	**Codes**
$G_{01(0a)(0b)(0d)} = [01 \cdots 01]$	$C_{01(0a)(0b)(0d)} = \{c_{i,i=0\ to\ 7}\}$
$G_{0v(0c)} = [0v \cdots 0v]$	$C_{0v(0c)} = \{c_0, c_2, c_4, c_6\}$
$G_{0v^2} = [0v^2 \cdots 0v^2]$	$C_{0v^2} = \{c_0, c_4\}$
$G_{10} = [10 \cdots 10]$	$C_{10} = \{c_0, c_1\}$
$G_{11(1a)(1b)(1d)} = [11 \cdots 11]$	$C_{11(1a)(1b)(1d)} = \{c_{i,i=0\ to\ 15}\}$
$G_{1v(1c)} = [1v \cdots 1v]$	$C_{1v(1c)} = \{c_{i,i=0,2,4,6,8,10,12,14}\}$
$G_{1v^2} = [1v^2 \cdots 1v^2]$	$C_{1v^2} = \{c_0, c_{12}\}$

where $\{c_0 = 00 \cdots 00,\ c_1 = 01 \cdots 01, c_2 = 0v \cdots 0v, c_3 = 0a \cdots 0a, c_4 = 0v^2 \cdots 0v^2, c_5 = 0b \cdots 0b, c_6 = 0c \cdots 0c, c_7 = 0d \cdots 0d, c_8 = 10 \cdots 10, c_9 = 11 \cdots 11, c_{10} = 1v \cdots 1v, c_{11} = 1a \cdots 1a, c_{12} = 1v^2 \cdots 1v^2, c_{13} = 1b \cdots 1b,\ c_{14} = 1c \cdots 1c, c_{15} = 1d \cdots 1d\}$.

The following theorems provide the covering radius of C_j, for $1 \leq j \leq 15$.

Theorem 3.1. Let $C_{j,1\leq j\leq 15}$ be the codes in the generator matric $G_{j,1\leq j\leq 15}$. Then

1. $\frac{3n}{4} \leq r_L(C_{01}) = r_L(C_{0a}) = r_L(C_{0b}) = r_L(C_{0d}) \leq \frac{5n}{2}$,
2. $n \leq r_L(C_{0v}) = r_L(C_{0c}) \leq 3n$,
3. $n \leq r_L(C_{0v^2}) \leq 3n$,
4. $2n \leq r_L(C_{10}) \leq 4n$,
5. $r_L(C_{11}) = r_L(C_{1a}) = r_L(C_{1b}) = r_L(C_{1d}) = 2n$,
6. $\frac{5n}{4} \leq r_L(C_{1v}) = r_L(C_{1c}) \leq \frac{3n}{2}$, and
7. $\frac{5n}{4} \leq r_L(C_{1v^2}) \leq 3n$, where $C_{j,1\leq j\leq 15}$ is the covering radius of codes with assigned to the lee weight in $\mathcal{R}$.

Proof.

1) For $c \in C_j, 1 \leq j \leq 15$, let $t_i(c), 0 \leq i \leq 15$ denote the number of occurrences of symbol i in the codeword c. Considering 1 to 15, that

$$r_L(C_j) = \max_{x\in\mathcal{R}^n}\{d_L(y, C_j): 1 \leq j \leq 15\}.$$

If $y \in \mathcal{R}^n$ and y is given $(t_0, t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8, t_9, t_{10}, t_{11}, t_{12}, t_{13}, t_{14}, t_{15})$, where $\sum_{j=0}^{15} t_i = n$, then

$$d_L(y, \overline{00}) = n - t_0 + t_2 + 3t_4 + t_6 + t_9 + 2t_{10} + t_{11} + 4t_{12} + t_{13} + 2t_{14} + t_{15},$$

$$
\begin{aligned}
d_L(y,\overline{01}) &= n - t_1 + t_3 + 3t_5 + t_7 + t_8 + t_{10} + 2t_{11} + t_{12} + 4t_{13} + t_{14} + 2t_{15},\\
d_L(y,\overline{0v}) &= n - t_2 + t_0 + t_4 + 3t_6 + t_9 + t_{11} + 2t_{12} + t_{13} + t_{15} + 4t_{14} + 2t_8,\\
d_L(y,\overline{0a}) &= n - t_3 + t_1 + t_5 + 3t_7 + t_8 + 2t_9 + t_{10} + t_{12} + 2t_{13} + t_{14} + 4t_{15},\\
d_L(y,\overline{0v^2}) &= n - t_4 + 3t_0 + t_2 + t_6 + 4t_8 + t_9 + 2t_{10} + t_{11} + t_{13} + 2t_{14} + t_{15},\\
d_L(y,\overline{0b}) &= n - t_5 + 3t_1 + t_3 + t_7 + t_8 + 4t_9 + t_{10} + 2t_{11} + t_{12} + t_{14} + 2t_{15},\\
d_L(y,\overline{0c}) &= n - t_6 + t_0 + 3t_2 + t_4 + 2t_8 + t_9 + 4t_{10} + t_{11} + 2t_{12} + t_{13} + t_{15},\\
d_L(y,\overline{0d}) &= n - t_7 + t_1 + 3t_3 + t_5 + t_8 + 2t_9 + t_{10} + 4t_{11} + t_{12} + 2t_{13} + t_{14},\\
d_L(y,\overline{10}) &= n - t_8 + t_1 + 2t_2 + t_3 + 4t_4 + t_5 + 2t_6 + t_7 + t_{10} + 3t_{12} + t_{14},\\
d_L(y,\overline{11}) &= n - t_9 + t_0 + t_2 + 2t_3 + t_4 + 4t_5 + t_6 + 2t_7 + t_{11} + 3t_{13} + t_{15},\\
d_L(y,\overline{1a}) &= n - t_{11} + t_0 + 2t_1 + t_2 + t_4 + 2t_5 + t_6 + 4t_7 + t_9 + t_{13} + 3t_{15},\\
d_L(y,\overline{1v^2}) &= n - t_{12} + 4t_0 + t_1 + 2t_2 + t_3 + t_5 + 2t_6 + t_7 + 3t_8 + t_9 + 3t_{14},\\
d_L(y,\overline{1b}) &= n - t_{13} + t_0 + 4t_1 + t_2 + 2t_3 + t_4 + t_6 + 2t_7 + 3t_9 + t_{11} + t_{15},\\
d_L(y,\overline{1c}) &= n - t_{14} + 2t_0 + t_1 + 4t_2 + t_3 + 2t_4 + t_5 + t_7 + t_8 + 3t_{10} + t_{12},\\
d_L(y,\overline{1d}) &= n - t_{15} + t_0 + 2t_1 + t_2 + 4t_3 + t_4 + 2t_5 + t_6 + t_9 + 3t_{11} + t_{13}.
\end{aligned}
$$

Therefore, $d_L(y,C_{01}) = d_L(y,C_{0a}) = d_L(y,C_{0b}) = d_L(y,C_{0d}) = min\{d_L(y,\overline{00}), d_L(y,\overline{01}),$ — $d_L(x,\overline{0v}), d_L(y,\overline{0a}), d_L(x,\overline{0v^2}), d_L(y,\overline{0b}), d_L(y,\overline{0c}), d_L(y,\overline{0d})\} \le \frac{5n}{2}$ and hence

$$r_L(C_{01}) = r_L(C_{0a}) = r_L(C_{0b}) = r_L(C_{0d}) \le \tfrac{5n}{2}.$$

If $y = (\overbrace{00\cdots00}^{\frac{n}{8}}\overbrace{01\cdots01}^{\frac{n}{8}}\overbrace{0v\cdots0v}^{\frac{n}{8}}\overbrace{0a\cdots0a}^{\frac{n}{8}}\overbrace{0v^2\cdots0v^2}^{\frac{n}{8}}\overbrace{0b\cdots0b}^{\frac{n}{8}}\overbrace{0c\cdots0c}^{\frac{n}{8}}\overbrace{0d\cdots0d}^{\frac{n}{8}}) \in \mathcal{R}^n$, then $d_L(y,C_{01}) = d_L(y,C_{0a}) = d_L(y,C_{0b}) = d_L(y,C_{0d}) = \frac{n}{16} + 2\left(\frac{n}{16}\right) + \frac{n}{16} + 4\left(\frac{n}{16}\right) + \frac{n}{16} + 2\left(\frac{n}{16}\right) + \frac{n}{16} = \frac{3n}{4}$. Thus $r_L(C_{01}) = r_L(C_{0a}) = r_L(C_{0b}) = r_L(C_{0d}) \ge \frac{3n}{4}$ and hence, $\frac{3n}{4} \le r_L(C_{01}) = r_L(C_{0a}) = r_L(C_{0b}) = r_L(C_{0d}) \le \frac{5n}{2}$.

2) In $C_{0v(0c)}$, $d_L(y,C_{0v}) = d_L(y,C_{0c}) = min\{d_L(y,\overline{00}), d_L(y,\overline{0v}), d_L(y,\overline{0v^2}), d_L(y,\overline{0c})\} \le 3n$. Thus, $r_L(C_{0v}) = r_L(C_{0c}) \le 3n$.

If $y = (\overbrace{00\cdots00}^{\frac{n}{4}}\overbrace{0v\cdots0v}^{\frac{n}{4}}\overbrace{0v^2\cdots0v^2}^{\frac{n}{4}}\overbrace{0c\cdots0c}^{\frac{n}{4}}) \in \mathcal{R}^n$, then $d_L(y,\overline{00}) = d_L(y,\overline{0v}) = d_L(y,\overline{0v^2}) = d_L(y,\overline{0c}) = 2\left(\frac{n}{8}\right) + 4\left(\frac{n}{8}\right) + 2\left(\frac{n}{8}\right) = n$. Thus $r_L(C_{0v}) = r_L(C_{0c}) \ge n$ and so, $n \le r_L(C_{0v}) = r_L(C_{0c}) \le 3n$.

3) In C_{0v^2}, $d_L(y,C_{0v^2}) = min\{d_L(y,\overline{00}), d_L(y,\overline{0v^2})\} \le 3n$ and hence $r_L(C_{0v^2}) \le 3n$.

If $y = (\overbrace{00\cdots00}^{\frac{n}{2}}\overbrace{0v^2\cdots0v^2}^{\frac{n}{2}}) \in \mathcal{R}^n$, then $d_L(y,\overline{00}) = d_L(y,\overline{0v^2}) = n$. Therefore, $r_L(C_{0v^2}) \ge n$ and thus $n \le r_L(C_{0v^2}) \le 3n$.

4) In C_{10}, $d_L(y, C_{10}) = min\{d_L(y, \overline{00}), d_L(y, \overline{01})\} \leq 4n$ then $r_L(C_{10}) \leq 4n$.

If $y = (\overbrace{00 \cdots 00}^{\frac{n}{2}} \overbrace{01 \cdots 01}^{\frac{n}{2}}) \in \mathcal{R}^n$, then $d_L(y, \overline{00}) = d_L(y, \overline{01}) = 2n$. Thus $r_L(C_{10}) \geq 2n$ and hence $2n \leq r_L(C_{10}) \leq 4n$.

5) In $C_{11(1a)(1b)(1d)}$, $d_L(y, C_{11}) = d_L(y, C_{1a}) = d_L(y, C_{1b}) = d_L(y, C_{1d}) = min\{d_L(y, \overline{00}), d_L(y, \overline{01}),$ $d_L(y, \overline{0v}), d_L(y, \overline{0a}), d_L(y, \overline{0v^2}), d_L(y, \overline{0b}), d_L(y, \overline{0c}), d_L(y, \overline{0d}), d_L(y, \overline{10}), d_L(y, \overline{11}), d_L(y, \overline{1v}),$ $d_L(y, \overline{1a}), d_L(y, \overline{1v^2}), d_L(y, \overline{1b}), d_L(y, \overline{1c}), d_L(y, \overline{1d})\} \leq 2n$. Therefore

$$r_L(C_{11}) = r_L(C_{1a}) = r_L(C_{1b}) = r_L(C_{1d}) \leq 2n.$$

Let $y = (\overbrace{00 \cdots 00}^{\frac{n}{16}} \overbrace{01 \cdots 01}^{\frac{n}{16}} \overbrace{0v \cdots 0v}^{\frac{n}{16}} \overbrace{0a \cdots 0a}^{\frac{n}{16}} \overbrace{0v^2 \cdots 0v^2}^{\frac{n}{16}} \overbrace{0b \cdots 0b}^{\frac{n}{16}} \overbrace{0c \cdots 0c}^{\frac{n}{16}} \overbrace{0d \cdots 0d}^{\frac{n}{16}} \overbrace{10 \cdots 10}^{\frac{n}{16}}$ $\overbrace{11 \cdots 11}^{\frac{n}{16}} \overbrace{1v \cdots 1v}^{\frac{n}{16}} \overbrace{1a \cdots 1a}^{\frac{n}{16}} \overbrace{1v^2 \cdots 1v^2}^{\frac{n}{16}} \overbrace{1b \cdots 1b}^{\frac{n}{16}} \overbrace{1c \cdots 1c}^{\frac{n}{16}} \overbrace{1d \cdots 1d}^{\frac{n}{16}}) \in \mathcal{R}^n$, then $d_L(y, \overline{00}) = d_L(y, \overline{01}) = d_L(y, \overline{0v}) = d_L(y, \overline{0a}) = d_L(y, \overline{0v^2}) = d_L(y, \overline{0b}) = d_L(y, \overline{0c}) = d_L(y, \overline{0d}) = d_L(y, \overline{10}) = d_L(y, \overline{11}) = d_L(y, \overline{1v}) = d_L(y, \overline{1a}) = d_L(y, \overline{1v^2}) = d_L(y, \overline{1b}) = d_L(y, \overline{1c}) = d_L(y, \overline{1d}) = \frac{n}{16} + 2\left(\frac{n}{16}\right) + \frac{n}{16} + 4\left(\frac{n}{16}\right) + \frac{n}{16} + 2\left(\frac{n}{16}\right) + \frac{n}{16} + \frac{n}{16} + 2\left(\frac{n}{16}\right) + 3\left(\frac{n}{16}\right) + 2\left(\frac{n}{16}\right) + 5\left(\frac{n}{16}\right) + 2\left(\frac{n}{16}\right) + 3\left(\frac{n}{16}\right) + 2\left(\frac{n}{16}\right) = \frac{32n}{16} = 2n$. Thus $r_L(C_{11}) = r_L(C_{1a}) = r_L(C_{1b}) = r_L(C_{1d}) \geq 2n$ and hence, $r_L(C_{11}) = r_L(C_{1a}) = r_L(C_{1b}) = r_L(C_{1d}) = 2n$.

6) In C_{1v}, $d_L(y, C_{1v}) = d_L(y, C_{1c}) = min\{d_L(y, \overline{00}), d_L(y, \overline{0v^2}), d_L(y, \overline{1v}), d_L(y, \overline{1c})\} \leq \frac{3n}{2}$, then

$$r_L(C_{1v}) = r_L(C_{1c}) \leq \frac{3n}{2}.$$

If $y = (\overbrace{00 \cdots 00}^{\frac{n}{4}} \overbrace{0v^2 \cdots 0v^2}^{\frac{n}{4}} \overbrace{1v \cdots 1v}^{\frac{n}{4}} \overbrace{1c \cdots 1c}^{\frac{n}{4}}) \in \mathcal{R}^n$, then $d_L(y, \overline{00}) = d_L(y, \overline{0v^2}) = d_L(y, \overline{1v}) = d_L(y, \overline{1c}) = 4\left(\frac{n}{8}\right) + 3\left(\frac{n}{8}\right) + 3\left(\frac{n}{8}\right) = \frac{5n}{4}$. Thus $r_L(C_{1v}) = r_L(C_{1c}) \geq \frac{5n}{4}$ and so, $\frac{5n}{4} \leq r_L(C_{1u}) = r_L(C_{1c}) \leq \frac{3n}{2}$.

7) In C_{1v^2}, $d_L(y, C_{1v^2}) = min\{d_L(y, \overline{00}), d_L(y, \overline{1v^2})\} \leq 3n$, then $r_L(C_{1v^2}) \leq 3n$.

If $y = (\overbrace{00 \cdots 00}^{\frac{n}{2}} \overbrace{1v^2 \cdots 1v^2}^{\frac{n}{2}}) \in \mathcal{R}^n$, then $d_L(y, \overline{00}) = d_L(y, \overline{1v^2}) = \frac{5n}{4}$. Thus $r_L(C_{1v^2}) \geq \frac{5n}{4}$ and hence $\frac{5n}{4} \leq r_L(C_{1v^2}) \leq 3n$.

Theorem 3.2. The covering radius of $C_{j,1 \leq j \leq 15}$, with respect to the Euclidean, Chinese Euclidean and Homogeneous weights are given by

Table 3. The Euclidean, Chinese Euclidean and Homogeneous weights.

Codes	Euclidean Weight	Chinese Euclidean Weight	Homogeneous Weight
$(C_{01(0a)(0b)(0d)}) = C_1$	$2n \leq r_E(C_1) \leq 5n$	$n \leq r_{CE}(C_1) \leq 3n$	$n \leq r_{Hom}(C_1) \leq 4n$
$(C_{0u(0c)}) = C_2$	$\frac{3n}{2} \leq r_E(C_2) \leq 6n$	$n \leq r_{CE}(C_2) \leq 3n$	$n \leq r_{Hom}(C_2) \leq 5n$
$(C_{0u^2}) = C_3$	$n \leq r_E(C_3) \leq 6n$	$n \leq r_{CE}(C_3) \leq 3n$	$n \leq r_{Hom}(C_3) \leq 4n$
$(C_{10}) = C_4$	$\frac{n}{4} \leq r_E(C_4) \leq 7n$	$\frac{n}{4} \leq r_{CE}(C_4) \leq 3n$	$\frac{n}{4} \leq r_{Hom}(C_4) \leq 5n$
$(C_{11(1a)(1b)(1d)}) = C_5$	$\frac{9n}{4} \leq r_E(C_5) \leq \frac{9n}{2}$	$\frac{5n}{4} \leq r_{CE}(C_5) \leq \frac{5n}{2}$	$\frac{5n}{4} \leq r_{Hom}(C_5) \leq 3n$
$(C_{1v(1c)}) = C_6$	$\frac{7n}{4} \leq r_E(C_6) \leq \frac{11n}{2}$	$\frac{5n}{4} \leq r_{CE}(C_6) \leq 4n$	$\frac{5n}{4} \leq r_{Hom}(C_6) \leq 5n$
$(C_{1v^2}) = C_7$	$\frac{5n}{4} \leq r_E(C_7) \leq \frac{11n}{2}$	$\frac{5n}{4} \leq r_{CE}(C_7) \leq \frac{5n}{2}$	$\frac{5n}{4} \leq r_{Hom}(C_7) \leq 3n$

Proof. Use to Theorem 3.1 with different weights such as Euclidean, Chinese Euclidean and Homogeneous. □

Block repetition code in $\mathcal{R}$

Let $C^n\colon BRep^{n_1+n_2+\cdots+n_{15}}$ be the block repetition code over $\mathcal{R}$ is an $\mathcal{R}$-additive code. Then the generator matrix $G = [\overbrace{0101\cdots01}^{n_1}\overbrace{0v0v\cdots0v}^{n_2}\overbrace{0a0a\cdots0a}^{n_3}\overbrace{0v^20v^2\cdots0v^2}^{n_4}\overbrace{0b0b\cdots0b}^{n_5}\overbrace{0c0c\cdots0c}^{n_6}$

$\overbrace{0d0d\cdots0d}^{n_7}\overbrace{1010\cdots10}^{n_8}\overbrace{1111\cdots11}^{n_9}\overbrace{1v1v\cdots1v}^{n_{10}}\overbrace{1a1a\cdots1a}^{n_{11}}\overbrace{1v^21v^2\cdots1v^2}^{n_{12}}\overbrace{1b1b\cdots0b}^{n_{13}}\overbrace{1c1c\cdots1c}^{n_{14}}$

$\overbrace{1d1d\cdots1d}^{n_{15}}]$.

The parameters of C^n:

$n = \sum_{j=1}^{15} n_j$,
$N = 16$,

$$d_L = min\{(32n_1 + 32n_2 + 24n_3 + 32n_4 + 24n_5 + 32n_6 + 24n_7 + 8n_8 + 32n_9 + 32n_{10} + 32n_{11} + 40n_{12} + 32n_{13} + 40n_{14} + 32n_{15}), 32(n_1 + n_2 + n_4 + n_6 + n_9 + n_{10} + n_{11} + n_{13} + n_{15}) + 24(n_3 + n_5 + n_7) + 8n_8 + 40(n_{12} + n_{14})\},$$

$$d_E = min\{(72n_1 + 48n_2 + 64n_3 + 32n_4 + 64n_5 + 48n_6 + 64n_7 + 8n_8 + 72n_9 + 56n_{10} + 72n_{11} + 56n_{12} + 64n_{13} + 56n_{14} + 72n_{15}), 72(n_1 + n_9 + n_{11} + n_{15}) + 48(n_2 + n_6) + 64(n_3 + n_5 + n_7) + 32n_4 + 8n_8 + 56(n_{10} + n_{12} + n_{14}) + 64n_{13}\},$$

$$d_{CE} = min\{(40n_1 + 32n_2 + 32n_3 + 32n_4 + 32n_5 + 32n_6 + 32n_7 + 8n_8 + 40n_9 + 40n_{10} + 40n_{11} + 40n_{12} + 40n_{13} + 40n_{14} + 40n_{15}), 40(n_1 + n_9 + n_{10} + n_{11} + n_{12} + n_{13} + n_{14} + n_{15}) + 32(n_2 + n_3 + n_4 + n_5 + n_6 + n_7) + 8n_8\},$$

$$d_{Hom} = min\{(40n_1 + 32n_2 + 32n_3 + 32n_4 + 32n_5 + 32n_6 + 32n_7 + 8n_8 + 40n_9 + 40n_{10} + 40n_{11} + 40n_{12} + 40n_{13} + 40n_{14} + 40n_{15}), 40(n_1 + n_9 + n_{10} + n_{11} + n_{12} + n_{13} + n_{14} + n_{15}) + 32(n_2 + n_3 + n_4 + n_5 + n_6 + n_7) + 8n_8\}.$$

Theorem 3.3. Let C^n be the block repetition code in $\mathcal{R}$ with length n. Then the covering radius of the block repetition codes are

1. $\frac{3(n_1+n_3+n_5+n_7)+4(n_2+n_4+n_6)+8(n_8+n_9+n_{11}+n_{13}+n_{15})+5(n_{10}+n_{12}+n_{14})}{4} \leq r_L(C^n) \leq$
   $\frac{40(n_1+n_3+n_5+n_7+n_{10}+n_{14})+48(n_2+n_4+n_6+n_8+n_{12})+32(n_9+n_{11}+n_{13}+n_{15})}{16}$,
2. $\frac{3(n_1+n_3+n_5+n_7)+6(n_2+n_4)+4n_4+n_8+9(n_9+n_{11}+n_{13}+n_{15})+7(n_{10}+n_{14})+5n_{12}}{4} \leq r_E(C^n) \leq$
   $\frac{80(n_1+n_3+n_5+n_7+n_9+n_{11}+n_{13}+n_{14}+n_{15})+62(n_2+n_{12})+96(n_4+n_6+n_{10})+144n_8}{16}$,
3. $\frac{4(n_1+n_2+n_3+n_4+n_5+n_6+n_7)+n_8+5(n_9+n_{10}+n_{11}+n_{12}+n_{13}+n_{14}+n_{15})}{4} \leq r_{CE}(C^n) \leq$
   $3(n_1+n_2+n_3+n_4+n_5+n_6+n_7+n_8+n_9+n_{10}+n_{11}+n_{12}+n_{13}+n_{14}+n_{15})$,
4. $\frac{4(n_1+n_2+n_3+n_4+n_5+n_6+n_7)+n_8+5(n_9+n_{10}+n_{11}+n_{12}+n_{13}+n_{14}+n_{15})}{4} \leq r_{Hom}(C^n) \leq$
   $3(n_1+n_8+n_{11}+n_{12}+n_{13}+n_{15})+4(n_2+n_3+n_4+n_5+n_6+n_7+n_9+n_{10}+n_{14})$.

Proof.
Use to ref. [13] and Theorem 3.1, 3.2, thus

$r_L(C^n) \geq \frac{3(n_1+n_3+n_5+n_7)+4(n_2+n_4+n_6)+8(n_8+n_9+n_{11}+n_{13}+n_{15})+5(n_{10}+n_{12}+n_{14})}{4}$.

Let $x = x_1x_2x_3x_4x_5x_6x_7x_8x_9x_{10}x_{11}x_{12}x_{13}x_{14}x_{15} \in \mathcal{R}^n$ with $x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}$, $x_{11}, x_{12}, x_{13}, x_{14}, x_{15}$ is $(a_i), (b_i), (c_i), (d_i), (e_i), (f_i), (g_i), (h_i), (k_i), (l_i), (m_i), (n_i), (o_i), (p_i)$, $(q_i)_{i=0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15}$, respectively such that $n_1 = \sum_{j=0}^{15} a_j, n_2 = \sum_{j=0}^{15} b_j$, $n_3 = \sum_{j=0}^{15} c_j$, $n_4 = \sum_{j=0}^{15} d_j$, $n_5 = \sum_{j=0}^{15} e_j$, $n_6 = \sum_{j=0}^{15} f_j$, $n_7 = \sum_{j=0}^{15} g_j$, $n_8 = \sum_{j=0}^{15} h_j$, $n_9 = \sum_{j=0}^{15} k_j$, $n_{10} = \sum_{j=0}^{15} l_j$, $n_{11} = \sum_{j=0}^{15} m_j, n_{12} = \sum_{j=0}^{15} n_j, n_{13} = \sum_{j=0}^{15} o_j, n_{14} = \sum_{j=0}^{15} p_j, n_{15} = \sum_{j=0}^{15} q_j$.

Thus, $r_L(C^n) \leq \frac{40(n_1+n_3+n_5+n_7+n_{10}+n_{14})+48(n_2+n_4+n_6+n_8+n_{12})+32(n_9+n_{11}+n_{13}+n_{15})}{16}$.

Similarly, $r_E(C^n), r_{CE}(C^n), r_{Hom}(C^n)$. □

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