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The Modular Irregularity Strength of $C_n \odot mK_1$

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Abstract

Let $G = (V, E)$ be a graph of order n with no component of order 2. An edge k-labeling $\alpha: E(G) \rightarrow \{1, 2, \dots, k\}$ is called a modular irregular k-labeling of graph G if the corresponding modular weight function $wt_\alpha: V(G) \rightarrow Z_n$ defined by $wt_\alpha(x) = \sum_{xy \in E(G)} \alpha(xy)$ is bijective. The value $wt_\alpha(x)$ is called the modular weight of vertex x. Minimum k such that G has a modular irregular k-labeling is called the modular irregularity strength of graph G. In this paper, we define modular irregular labeling of $C_n \odot mK_1$. Furthermore, we determine the modular irregularity strength of $C_n \odot mK_1$.
Keywords: corona product; cycle; empty graph; modular irregular labeling; modular irregularity strength.

Abstrak

Diberikan graf $G = (V, E)$ dengan orde n dengan tidak ada komponen yang berorde 2. Sebuah pelabelan-k sisi $\alpha: E(G) \rightarrow \{1, 2, \dots, k\}$ disebut pelabelan-k tak teratur modular pada graf G jika fungsi bobot modularnya $wt_\alpha : V(G) \rightarrow Z_n$ *dengan $wt_\alpha(x) = \sum_{xy \in E(G)} \alpha(xy)$ merupakan fungsi bijektif. Nilai $wt_\alpha(x)$ disebut bobot modular dari simpul x. Minimum dari k sehingga G mempunyai pelabelan-k tak teratur modular disebut dengan kekuatan ketakteraturan modular dari graf G. Pada tulisan ini, didefinisikan pelabelan tak teratur modular pada $C_n \odot mK_1$. Lebih lanjut, ditentukan kekuatan ketakteraturan modular dari $C_n \odot mK_1$*
Kata Kunci: *hasil kali korona*; *lingkaran*, *graf kosong*; *pelabelan tak teratur modular*; *kekuatan ketakteraturan modular.*

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1. INTRODUCTION

Let G be a graph. *Labeling* of graph G is a function that assigns a set of graph elements to a set of numbers. Research on graph labeling was initiated by Rossa in 1967 [1]. Some types of labeling that have been widely studied are graceful labeling, magic labeling, antimagic labeling, and irregular labeling.

The idea of irregular labeling was first proposed by Chartrand et al. in [2]. If on a simply connected graph G there exists an edge k-labeling such that its vertex weights are distinct, then the k-labeling is called an *irregular k-labeling* of G. The problem studied in irregular labeling is determining the minimum value k of any irregular k-labeling of graph G. This parameter is called the *irregularity strength* of G, denoted by $s(G)$. An upper bound for the irregularity strength of the graph is given by Kalkowski et al. in [3]. Some findings of the irregular assignment can be seen in [4]–[6].

There are some types of labelings motivated by this labeling, namely edge irregular labeling, vertex irregular total labeling, edge irregular total labeling, etc. An *edge irregular k-labeling* is a vertex k-labeling whose edge weights are distinct. Ahmad et al. gave a lower bound of the edge irregularity strength

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in [7]. Several results of the edge irregularity strength of graphs can be seen in [7]–[10]. Encouraged by the idea of various kinds of other total labelings, total edge irregular labeling, and total vertex irregular labeling can be defined. The total edge irregularity strength of some graphs is determined in [11]–[18]. While some findings on the total vertex irregularity strength of graphs are shown in [19]–[22].

Another type of labeling inspired by irregular labeling is modular irregular labeling which is introduced by Bača et al. in [23]. Let $G = (V, E)$ be a graph of order n with no component of order 2. An edge k-labeling $\alpha: E(G) \longrightarrow \{1, 2, \dots, k\}$ is called a *modular irregular k-labeling* of G if the corresponding weight function $wt_\alpha(x): V(G) \longrightarrow Z_n$ defined by

$$wt_\alpha(x) = \sum_{xy \in E(G)} \alpha(xy)$$

is bijective. The value $wt_\alpha(x)$ is called the *modular weight* of the vertex x and Z_n is a complete set of residues modulo n.

The minimum value of k such that G has a modular irregular k-labeling is called the *modular irregularity strength* of G, denoted by $ms(G)$. If there is no modular irregular labeling of G, we define $ms(G) = \infty$. In [23], Bača et al. proposed some fundamental lemmas and theorems about modular irregular labeling as follows.

Lemma 1. [23] Let $G = (V, E)$ be a graph with no component of order 2. Any modular irregular k-labeling of G is also an irregular labeling of G.

Lemma 2. [23] *Let* $G = (V, E)$ be a graph with no component of order ≤ 2 and $s(G) = k$. If there exists an irregular k-labeling of G such that weights of the vertices constitute a set of consecutive integers, then

$$s(G) = ms(G) = k.$$

Theorem 1. [23] Let $G = (V, E)$ be a graph with no component of order ≤ 2. Then

$$s(G) \leq ms(G).$$

Theorem 2. [23] If $G = (V, E)$ is a graph of order n, $n \equiv 2 \ (mod\ 4)$, then G has no modular irregular k-labeling, i.e. $ms(G) = \infty$.

In [23], Bača et al. proved that some graphs have modular irregular labelings, such as path, star, triangular graph, gear, and cycle. The modular irregularity strength of complete graphs and some classes of bipartite graphs are proved in [24]. Tilukay in [25] proved that triangular book graphs admit a modular irregular labeling and its modular irregularity strength and irregularity strength are equal, except for a small case and the infinity property. While, Sugeng et al. in [26] defined modular irregular labelings of the regular double-star graphs and friendship graphs. Hinding et al. proved the modular irregularity strength of dodecahedral-modified generalization graphs in [27]. Furthermore, in [28], Muthugurupackiam and Ramya proved the modular irregularity strength of two classes of graphs.

Considering Theorem 1, the interesting topic is the conditions for a graph G such that $s(G) = ms(G)$ and $s(G) < ms(G)$. Bača et al. in [29] compared the irregularity strength and modular irregularity strength of wheels. If $n = 5$ or $n \equiv 1 \ (mod\ 4)$, then $s(W_n) < ms(W_n)$, otherwise $s(W_n) = ms(W_n)$. In [30], Bača et al. proved that the irregularity strength and modular irregularity strength of fan graphs are not always the same. If $n = 8$ or $n \equiv 1\ (mod\ 4)$, then $s(F_n) < ms(F_n)$, otherwise $s(F_n) = ms(F_n)$. While in [31], Apituley et al. proved that the irregularity strength of friendship graphs is equal to its modular irregularity strength.

The corona product of two graphs G and H, denoted by $G \odot H$, is a graph obtained by taking one copy of G (which has n vertices) and n copies $H_1, H_2, \dots, H_n$ of H, and then joining the i-th vertex of G to every vertex in H_i. In [32], Muthugurupackiam and Ramya defined modular irregular labeling of corona products of C_m and P_n for $1 \leq n \leq 3$ and also determined its modular irregularity strength. In [33], it was defined as irregular labeling of corona product of C_n and mK_1, denoted by $C_n \odot mK_1$. The order of $C_n \odot mK_1$ is $(m+1)n$; since it has a copy of C_n and n copies of mK_1. The vertices' weights of $C_n \odot mK_1$ in [33] do not constitute a set of consecutive integers nor form a complete set of residues of modulo $(m+1)n$. Furthermore, it was proved that $s(C_n \odot mK_1) = mn$. Later in this paper, we define modular irregular labeling of $C_n \odot mK_1$ and determine its modular irregularity strength.

2. METHODS

We provided definitions, lemmas, and theorems of modular irregular labeling based on some references. We define the notations of the vertices of $C_n \odot mK_1$. Then, we construct an edge mn-labeling of $C_n \odot mK_1$ such that the vertices' weights constitute a complete set of residues of modulo $(m+1)n$. The last, we determine its modular irregularity strength by considering Theorem 1.

3. RESULTS AND DISCUSSION

According to Theorem 2, modular irregular labeling is not able to be defined of $C_n \odot mK_1$ if its order is congruent to 2(mod 4). Thus, graphs $C_n \odot mK_1$ which n and m as given in Table 1 do not have modular irregular labeling.

Table 1. The values n and m for which $C_n \odot mK_1$ has no irregular modular labeling

n	m
1 (mod 4)	1 (mod 4)
2 (mod 4)	0 (mod 4)
2 (mod 4)	2 (mod 4)
3 (mod 4)	1 (mod 4)

Here, we define a modular irregular mn-labeling of $C_n \odot mK_1$ of order $(m+1)n$, $(m+1)n \not\equiv 2(mod\ 4)$ and prove that its modular irregularity strength is equal to its irregularity strength.

Theorem 3. For $n \geq 3$ and $m \geq 1$, let $C_n \odot mK_1$ be a corona product of C_n and mK_1 of order $(m+1)n$ where $(m+1)n \not\equiv 2\ (mod\ 4)$. Then $ms(C_n \odot mK_1) = mn$.

Proof.
First, we define the notation of each vertex in $C_n \odot mK_1$. Let the vertices of the cycle C_n be denoted by $\{u_1, u_2, \dots, u_n\}$ and the vertices of mK_1 which correspond to vertex $u_i \in C_n$ are denoted by $\{v_i^1, v_i^2, \dots, v_i^m\}$. The modular irregular mn-labeling of $C_n \odot mK_1$ is divided into six cases.

<u>Case 1: $n \equiv 1(mod\ 2)$ and $m \equiv 3(mod\ 4)$; $n \equiv 0(mod\ 2)$ and $m \equiv 1(mod\ 4)$</u>

For the first case, we define an edge mn-labeling $\alpha: E(C_n \odot mK_1) \longrightarrow \{1, 2, \dots, mn\}$ as follows. For $1 \leq i \leq n$ and $1 \leq j \leq m$,

$$\alpha(u_i v_i^j) = \begin{cases} n(j-1)+i \text{ ; if } j \text{ is odd} \\ nj - i + 1 \text{ ; if } j \text{ is even} \end{cases}, \quad (1)$$

$$\alpha(u_i u_{i+1}) = \alpha(u_n u_1) = \frac{m(2n-1)+1}{4}.$$

Hence, the weights of the vertices are as follows.
For $1 \leq i \leq n$ and $1 \leq j \leq m$,

$$wt_\alpha(v_i^j) = \alpha(u_i v_i^j) = \begin{cases} n(j-1)+i \text{ ; if } j \text{ is odd} \\ nj - i + 1 \text{ ; if } j \text{ is even} \end{cases}, \quad (2)$$

$$wt_\alpha(u_i) = mn + i + \frac{m-1}{2} n(m+1) \equiv (mn+i)\left(mod\ n(m+1)\right).$$

The vertices' weights given in (2) constitute a complete set of residues modulo $n(m+1)$. Thus, the edge mn-labeling given in (1) is a modular irregular mn-labeling of $C_n \odot mK_1$. In Figure 1, we show a modular irregular 9-labeling of $C_3 \odot 3K_1$ and its modular vertices' weights.

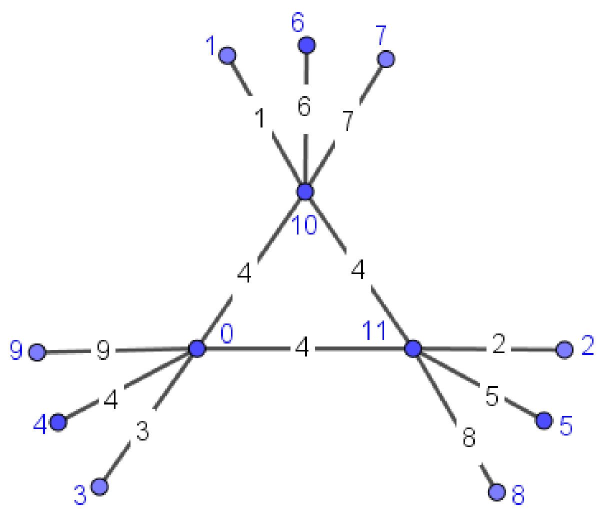

Figure 1. A modular irregular 9-labeling of $C_3 \odot 3K_1$ and its modular vertices' weights.

<u>Case 2: $n \equiv 1(mod\ 4)$ and $m \equiv 0(mod\ 2)$</u>

For the second case, we define an edge mn-labeling $\alpha: E(C_n \odot mK_1) \longrightarrow \{1, 2, \dots, mn\}$ as follows. For $1 \leq i \leq n$ and $1 \leq j \leq m$,

$$\alpha\left(u_i v_i^j\right) = \begin{cases} n(j-1)+i & ;\text{if } j \text{ is odd} \\ nj-i+1 & ;\text{if } j \text{ is even, } j<m \\ n(m-1)+i-\frac{n-1}{2} & ; j=m,\ i>\frac{n}{2} \\ n(m-1)+i+\frac{n+1}{2} & ; j=m,\ i<\frac{n}{2} \end{cases}, \tag{3}$$

$$\alpha(u_i u_{i+1}) = \alpha(u_n u_1) = \frac{(n-1)(m-1)}{4}.$$

Hence, the weights of the vertices are as follows.
For $1 \le i \le n$ and $1 \le j \le m$,

$$wt_\alpha\left(v_i^j\right) = \alpha\left(u_i v_i^j\right) = \begin{cases} n(j-1)+i & ;\text{if } j \text{ is odd} \\ nj-i+1 & ;\text{if } j \text{ is even, } j<m \\ n(m-1)+i-\frac{n-1}{2} & ; j=m,\ i>\frac{n}{2} \\ n(m-1)+i+\frac{n+1}{2} & ; j=m,\ i<\frac{n}{2} \end{cases}, \tag{4}$$

For $1 \le i < \frac{n}{2}$, $wt_\alpha(u_i) = \frac{m}{2}n(m+1) + 2i - n \equiv (nm+2i)\left(mod\, n(m+1)\right)$.

For $\frac{n}{2} < i \le n$, $wt_\alpha(u_i) = \frac{m}{2}n(m+1) + 2i - 2n \equiv (nm-n+2i)\left(mod\, n(m+1)\right)$.

The vertices' weights given in (4) constitute a complete set of residues modulo $n(m+1)$. Thus, the edge mn-labeling given in (3) is a modular irregular mn-labeling of $C_n \odot mK_1$. In Figure 2, we show a modular irregular 20-labeling of $C_5 \odot 4K_1$ and its modular vertices' weights.

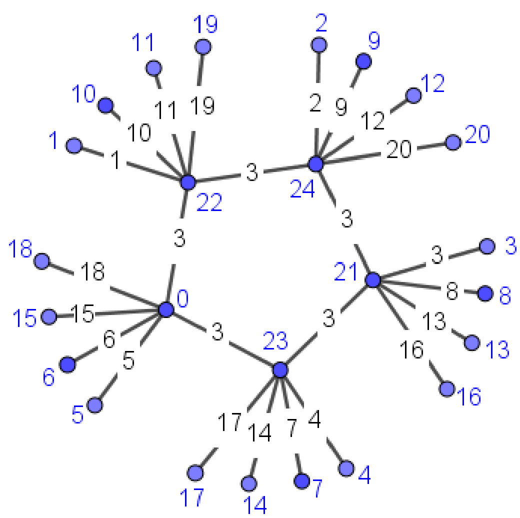

Figure 2. A modular irregular 20-labeling of $C_5 \odot 4K_1$ and its modular vertices' weights.

<u>Case 3: $n \equiv 3(mod\ 4)$ and $m \equiv 0(mod\ 2)$</u>

For the third case, we define an edge mn-labeling $\alpha: E(C_n \odot mK_1) \longrightarrow \{1, 2, \dots, mn\}$ as follows.
For $1 \le i \le n$ and $1 \le j \le m$,

$$\alpha(u_i v_i^j) = \begin{cases} n(j-1)+i & ;\text{if } j \text{ is odd} \\ nj-i+1 & ;\text{if } j \text{ is even, } j<m \\ n(m-1)+i-\frac{n-1}{2} & ;j=m,\ i>\frac{n}{2} \\ n(m-1)+i+\frac{n+1}{2} & ;j=m,\ i<\frac{n}{2} \end{cases}, \tag{5}$$

$$\alpha(u_i u_{i+1}) = \alpha(u_n u_1) = n + \frac{(3n-1)(m-1)}{4}.$$

Hence, the weights of the vertices are as follows.
For $1 \le i \le n$ and $1 \le j \le m$,

$$wt_\alpha(v_i^j) = \alpha(u_i v_i^j) = \begin{cases} n(j-1)+i & ;\text{if } j \text{ is odd} \\ nj-i+1 & ;\text{if } j \text{ is even, } j<m \\ n(m-1)+i-\frac{n-1}{2} & ;j=m,\ i>\frac{n}{2} \\ n(m-1)+i+\frac{n+1}{2} & ;j=m,\ i<\frac{n}{2} \end{cases}, \tag{6}$$

For $1 \le i < \frac{n}{2}$, $wt_\alpha(u_i) = nm + 2i + n(m+1)\frac{m}{2} \equiv (nm+2i)\left(mod\ n(m+1)\right)$.

For $\frac{n}{2} < i \le n$, $wt_\alpha(u_i) = nm + 2i - n + n(m+1)\frac{m}{2} \equiv (nm - n + 2i)\left(mod\ n(m+1)\right)$.

The vertices' weights given in (6) constitute a complete set of residues modulo $n(m+1)$. Thus, the edge mn-labeling given in (5) is a modular irregular mn-labeling of $C_n \odot mK_1$.

<u>Case 4: $n \equiv 0(mod\ 2)$ and $m \equiv 3(mod\ 4)$</u>

For the fourth case, we define an edge mn-labeling $\alpha: E(C_n \odot mK_1) \longrightarrow \{1, 2, \dots, mn\}$ as follows.
For $1 \le i \le n$ and $1 \le j \le m$

$$\alpha(u_i v_i^j) = \begin{cases} n(j-1)+i\ ; \text{if } j \text{ is odd} \\ nj-i+1\ ; \text{if } j \text{ is even} \end{cases}, \tag{7}$$

$$\alpha(u_i u_{i+1}) = \begin{cases} \frac{m(2n-1)-1}{4}, \text{if } i \text{ is odd} \\ \frac{m(2n-1)+3}{4}, \text{if } i \text{ is even}, i<n \end{cases},$$

$$\alpha(u_n u_1) = \frac{m(2n-1)+3}{4}.$$

Hence, the weights of the vertices are as follows.
For $1 \le i \le n$ and $1 \le j \le m$

$$wt_\alpha(v_i^j) = \alpha(u_i v_i^j) = \begin{cases} n(j-1)+i\ ; \text{if } j \text{ is odd} \\ nj-i+1\ ; \text{if } j \text{ is even} \end{cases}, \tag{8}$$

$$wt_\alpha(u_i) = mn + j + \frac{m-1}{2} n(m+1) \equiv (nm+i)\left(mod\ n(m+1)\right).$$

The vertices' weights given in (8) constitute a complete set of residues modulo $n(m+1)$. Thus, the edge mn-labeling given in (7) is a modular irregular mn-labeling of $C_n \odot mK_1$.

Case 5: $m, n \equiv 0(mod\ 4)$

For the fifth case, we define an edge mn-labeling $\alpha: E(C_n \odot mK_1) \longrightarrow \{1, 2, \dots, mn\}$ as follows. For $1 \le i \le n$ and $1 \le j \le m$,

$$\alpha(u_i v_i^j) = \begin{cases} n(j-1)+i\,; \text{if } j \text{ is odd} \\ nj-i+1\,; \text{if } j \text{ is even} \end{cases}, \tag{9}$$

$$\alpha(u_i u_{i+1}) = \begin{cases} \frac{m(3n-1)}{4}+i & ; 1 \le i < \frac{n}{2} \\ \frac{m(3n-1)}{4}+2\left\lceil\frac{n-i}{2}\right\rceil; & \frac{n}{2} \le i < n \end{cases},$$

$$\alpha(u_n u_1) = \frac{m(3n-1)}{4}.$$

Hence, the weights of the vertices are as follows.
For $1 \le i \le n$ and $1 \le j \le m$,

$$wt_\alpha(v_i^j) = \alpha(u_i v_i^j) = \begin{cases} n(j-1)+i\,; \text{if } j \text{ is odd} \\ nj-i+1\,; \text{if } j \text{ is even} \end{cases}, \tag{10}$$

$$wt_\alpha(u_i) = \begin{cases} \frac{mn(m+3)}{2}+2i-1 \equiv (mn+2i-1)\left(mod\ n(m+1)\right) & ; 1 \le i \le \frac{n}{2} \\ \frac{mn(m+3)}{2}+2(n-i+1) \equiv \left(mn+2(n-i+1)\right)\left(mod\ n(m+1)\right); & \frac{n}{2} < i \le n \end{cases}.$$

The vertices' weights given in (10) constitute a complete set of residues modulo $n(m+1)$. Thus, the edge mn-labeling given in (9) is a modular irregular mn-labeling of $C_n \odot mK_1$. In Figure 3, we show a modular irregular 16-labeling of $C_4 \odot 4K_1$ and its modular vertices' weights.

Case 6: $n \equiv 0(mod\ 4)$ and $m \equiv 2(mod\ 4)$

For the sixth case, we define an edge mn-labeling $\alpha: E(C_n \odot mK_1) \longrightarrow \{1, 2, \dots, mn\}$ as follows. For $1 \le i \le n$ and $1 \le j \le m$,

$$\alpha(u_i v_i^j) = \begin{cases} n(j-1)+i\,; \text{if } j \text{ is odd} \\ nj-i+1\,; \text{if } j \text{ is even} \end{cases}, \tag{11}$$

$$\alpha(u_i u_{i+1}) = \begin{cases} \frac{m(3n-1)}{4}+\frac{1}{2}+2\left\lceil\frac{i-1}{2}\right\rceil; & 1 \le i < \frac{n}{2} \\ \frac{m(3n-1)}{4}+\frac{1}{2}+n-i\,; & \frac{n}{2} \le i < n \end{cases},$$

$$\alpha(u_n u_1) = \frac{m(3n-1)}{4}+\frac{1}{2}.$$

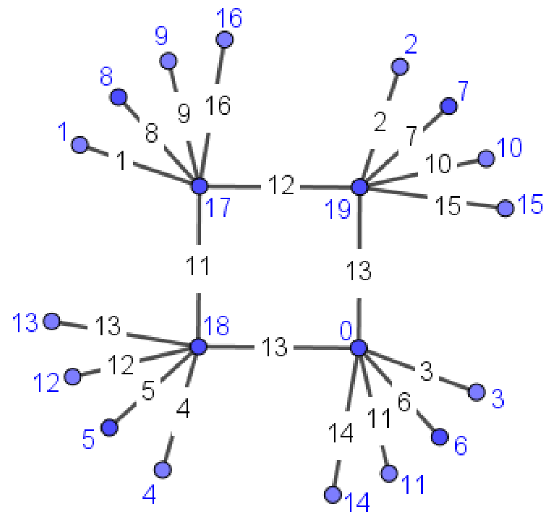

Figure 3. A modular irregular 16-labeling of $C_4 \odot 4K_1$ and its modular vertices' weights.

Hence, the weights of the vertices are as follows.
For $1 \le i \le n$ and $1 \le j \le m$,

$$wt_\alpha(v_i^j) = \alpha(u_i v_i^j) = \begin{cases} n(j-1)+i \text{ ; if } j \text{ is odd} \\ nj - i + 1 \text{ ; if } j \text{ is even} \end{cases}, \tag{12}$$

$$wt_\alpha(u_i) = \begin{cases} \frac{mn(m+3)}{2} + 2i - 1 \equiv (mn + 2i - 1)\left(mod\ n(m+1)\right) & ; 1 \le i \le \frac{n}{2} \\ \frac{mn(m+3)}{2} + 2(n-i+1) \equiv \left(mn + 2(n-i+1)\right)\left(mod\ n(m+1)\right); & \frac{n}{2} < i \le n \end{cases}.$$

The vertices' weights given in (12) constitute a complete set of residues modulo $n(m+1)$. Thus, the edge mn-labeling given in (11) is a modular irregular mn-labeling of $C_n \odot mK_1$. In each labeling (1), (3), (5), (7), (9), and (11), the maximum label is mn. From [33], we know that $s(C_n \odot mK_1) = mn$. Considering Theorem 1, we conclude that $s(C_n \odot mK_1) = mn$. ■

4. CONCLUSIONS

We defined modular irregular mn-labelings of $C_n \odot mK_1$ of order $(m+1)n$, $(m+1)n \not\equiv 2(mod\ 4)$, in six cases. We proved that $ms(C_n \odot mK_1) = nm$ if $(m+1)n \not\equiv 2(mod\ 4)$. Thus, its modular irregularity strength is equal to its irregularity strength except when $(m+1)n \not\equiv 2(mod\ 4)$.

REFERENCES

[1] J. A. Galian, "A dynamic survey of graph labeling," *The Electronic Journal of Combinatorics*, vol. #DS6, 2020.
[2] G. Chartrand, M. S. Jacobson, J. Lehel, O. R. Oellermann, S. Ruiz, and F. Saba, "Irregular networks," *In Congr. Numer*, vol. 64, pp. 197–210, 1988.

[3] M. Kalkowski, M. Karoński, and F. Pfender, "A new upper bound for the irregularity strength of graphs," *SIAM J Discret Math*, vol. 25, no. 3, pp. 1319–1321, Jan. 2011, doi: 10.1137/090774112.

[4] Nurdin, "Irregular assignment of series parallel networks," *J Phys Conf Ser*, vol. 979, p. 012070, Mar. 2018, doi: 10.1088/1742-6596/979/1/012070.

[5] M. Anholcer and C. Palmer, "Irregular labelings of circulant graphs," *Discrete Math*, vol. 312, no. 23, pp. 3461–3466, Dec. 2012, doi: 10.1016/j.disc.2012.06.017.

[6] P. Majerski and J. Przybyło, "On the irregularity strength of dense graphs," *SIAM J Discret Math*, vol. 28, no. 1, pp. 197–205, Jan. 2014, doi: 10.1137/120886650.

[7] A. Ahmad, O. B. S. Al-Mushayt, and M. Bača, "On edge irregularity strength of graphs," *Appl Math Comput*, vol. 243, pp. 607–610, Sep. 2014, doi: 10.1016/j.amc.2014.06.028.

[8] I. Tarawneh, R. Hasni, and A. Ahmad, "On the edge irregularity strength of corona product of cycle with isolated vertices," *AKCE International Journal of Graphs and Combinatorics*, vol. 13, no. 3, pp. 213–217, Dec. 2016, doi: 10.1016/j.akcej.2016.06.010.

[9] M. Imran, A. Aslam, S. Zafar, and W. Nazeer, "Further results on edge irregularity strength of graphs," *Indonesian Journal of Combinatorics*, vol. 1, no. 2, p. 36, Aug. 2017, doi: 10.19184/ijc.2017.1.2.5.

[10] A. Ahmad, M. Bača, and M. F. Nadeem, "On edge irregularity strength of Toeplitz graphs," *U.P.B. Sci. Bull., Series A*, vol. 78, 2016.

[11] L. Ratnasari and Y. Susanti, "Total edge irregularity strength of ladder-related graphs," *Asian-European Journal of Mathematics*, vol. 13, no. 04, p. 2050072, Jun. 2020, doi: 10.1142/S1793557120500722.

[12] P. Jeyanthi and A. Sudha, "Total edge irregularity strength of disjoint union of wheel graphs," *Electron Notes Discrete Math*, vol. 48, pp. 175–182, Jul. 2015, doi: 10.1016/j.endm.2015.05.026.

[13] N. Hinding, N. Suardi, and H. Basir, "Total edge irregularity strength of subdivision of star," *Journal of Discrete Mathematical Sciences and Cryptography*, vol. 18, no. 6, pp. 869–875, Nov. 2015, doi: 10.1080/09720529.2015.1032716.

[14] I. Rajasingh and S. T. Arockiamary, "Total edge irregularity strength of series parallel graphs," *International Journal of Pure and Apllied Mathematics*, vol. 99, no. 1, Feb. 2015, doi: 10.12732/ijpam.v99i1.2.

[15] R. W. Putra and Y. Susanti, "On total edge irregularity strength of centralized uniform theta graphs," *AKCE International Journal of Graphs and Combinatorics*, vol. 15, no. 1, pp. 7–13, Apr. 2018, doi: 10.1016/j.akcej.2018.02.002.

[16] D. Indriati, I. E. W. Widodo, K. A. Sugeng, and M. Bača, "On total edge irregularity strength of generalized web graphs and related graphs," *Mathematics in Computer Science*, vol. 9, no. 2, pp. 161–167, Jun. 2015, doi: 10.1007/s11786-015-0221-5.

[17] M. Bača and M. K. Siddiqui, "Total edge irregularity strength of generalized prism," *Appl Math Comput*, vol. 235, pp. 168–173, May 2014, doi: 10.1016/j.amc.2014.03.001.

[18] D. Indriati, I. E. W. Widodo, and K. A. Sugeng, "On the total edge irregularity strength of generalized helm," *AKCE International Journal of Graphs and Combinatorics*, vol. 10, no. 2, pp. 147–155, 2013.

[19] P. Jeyanthi and A. Sudha, "Total vertex irregularity strength of corona product of some graphs," *Journal of Algorithms and Computation*, vol. 48, no. 1, pp. 127–140, 2016.

[20] M. Imran, A. Ahmad, M. K. Siddiqui, and T. Mehmood, "Total vertex irregularity strength of generalized prism graphs," *Journal of Discrete Mathematical Sciences and Cryptography*, vol. 25, no. 6, pp. 1855–1865, Aug. 2022, doi: 10.1080/09720529.2020.1848103.
[21] A. Ahmad, M. Bača, and Y. Bashir, "Total vertex irregularity strength of certain classes of unicyclic graphs," *Bulletin mathématique de la Société des Sciences Mathématiques de Roumanie*, pp. 147–152, Jan. 2014.
[22] P. Majerski and J. Przybyło, "Total vertex irregularity strength of dense graphs," *J Graph Theory*, vol. 76, no. 1, pp. 34–41, May 2014, doi: 10.1002/jgt.21748.
[23] M. Bača, K. Muthugurupackiam, K. M. Kathiresan, and S. Ramya, "Modular irregularity strength of graphs," *Electronic Journal of Graph Theory and Applications*, vol. 8, no. 2, pp. 435–443, 2020, doi: 10.5614/ejgta.2020.8.2.19.
[24] K. W. Prasancika, "Kekuatan ketidakteraturan modular beberapa graf padat (english translation: The modular irregularity strength of some dense graphs)," Undergraduated Thesis, Universitas Pendidikan Ganesha, Singaraja, 2021.
[25] M. I. Tilukay, "Modular irregularity strength of triangular book graph," Nov. 2021, Accessed: Jul. 23, 2022. [Online]. Available: https://arxiv.org/ftp/arxiv/papers/2111/2111.12897.pdf
[26] K. A. Sugeng, Z. Z. Barack, N. Hinding, and R. Simanjuntak, "Modular irregular labeling on double-star and friendship graphs," *Journal of Mathematics*, vol. 2021, pp. 1–6, Dec. 2021, doi: 10.1155/2021/4746609.
[27] N. Hinding, K. A. Sugeng, . N., T. J. Wahyudi, and R. Simanjuntak, "Two types irregular labelling on dodecahedral modified generalization graph," *SSRN Electronic Journal*, 2021, doi: 10.2139/ssrn.3968029.
[28] K. Muthugurupackiam and S. Ramya, "Modular irregularity strength of two classes of graphs," *Journal of Computer and Mathematical Sciences*, vol. 9, no. 9, pp. 1132–1141, 2018.
[29] M. Bača, M. Imran, and A. Semaničová-Feňovčíková, "Irregularity and modular irregularity strength of wheels," *Mathematics*, vol. 9, no. 21, Nov. 2021, doi: 10.3390/math9212710.
[30] M. Bača, Z. Kimáková, M. Lascsáková, and A. Semaničová-Feňovčíková, "The irregularity and modular irregularity strength of fan graphs," *Symmetry (Basel)*, vol. 13, no. 4, p. 605, Apr. 2021, doi: 10.3390/sym13040605.
[31] F. A. N. J. Apituley, M. W. Talakua, and Y. A. Lesnussa, "On the irregularity strength and modular irregularity strength of friendship graphs and its disjoint union," *BAREKENG: Jurnal Ilmu Matematika dan Terapan*, vol. 16, no. 3, pp. 869–876, Sep. 2022, doi: 10.30598/barekengvol16iss3pp869-876.
[32] K. Muthugurupackiam and S. Ramya, "Modular labelings on some classes of graphs," *Aryabhatta Journal of Mathematics & Informatics*, vol. 12, no. 2, pp. 165–172, 2020.
[33] K. Muthugurupackiam, T. Manimaran, and A. Thuraiswamy, "Irregularity strength of corona of two graphs," *Lecture Notes in Comput. Sci.*, pp. 175–181, 2017.