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Statistical Modeling using A New Hybrid Form of The Inverted Exponential Distribution with Different Estimation Methods

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Abstract

This paper introduces a new four-parameter distribution called the exponentiated Gompertz generated inverted exponential (EGG_{IE}) distribution. Explicit expressions of the structural properties such as the ordinary and incomplete moments, probability weighted moments, quantile function, Lorenz and Bonferroni curves, entropies, and order statistics are derived. The empirical findings indicate that the maximum likelihood procedure dominates the other estimators in the simulation study while the Cramer-Von Mises procedure dominates in the two real datasets applications. We demonstrate the superiority of the EGG_{IE} distribution over the Gompertz Lomax, odd Fréchet Inverse exponential, generalized inverse exponential, generalized inverse exponential, exponential inverse exponential, and Gompertz Weibull distribution using the maximum likelihood procedure utilizing two real datasets applications. The findings show that the EGG_{IE} distribution yields the best goodness of fit to the two datasets.

Keywords: exponentiated Gompertz generated family; inverse exponential distribution; Kolmogorov-Smirnov statistic; Anderson-Darling; maximum product spacing.

Abstrak

Paper ini memperkenalkan distribusi 4-parameter baru yang disebut dengan distribusi exponentiated Gompertz generated inverted exponential (EGGIE). *Ekspresi eksplisit sifat struktural dari distribusi ini diturunkan, seperti momen biasa dan momen tak lengkap, momen probabilitas terboboti, fungsi kuartil, kurva* Lorenz *dan* Bonferroni, *entropi, dan statistik urutan. Temuan empiris menunjukan bahwa prosedur maksimum likelihood mendominasi estimator lainnya pada studi simulasi, sementara prosedur* Cramer-Von Mises *mendominasi pada aplikasi dua dataset nyata. Peneliti menunjukkan keunggulan dari distribusi* EGGIE *dibandingkan distribusi* Gompertz Lomax, odd Frechet Inverse exponential, generalized inverse exponential, exponential inverse exponential, *dan* Gompertz Weibull *menggunakan metode maksimum likelihood yang diaplikasikan pada dua dataset nyata. Hasil menunjukan bahwa distribusi* EGGIE *menghasilkan kecocokan model yang baik pada kedua dataset.*

Kata Kunci: *keluarga bangkitan* exponentiated Gompertz; *distribusi* inverse exponential; Kolmogorov-Smirnov statistic; Anderson-Darling; maximum product spacing.

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1. INTRODUCTION

Keller and Kamath [1] introduced the inverted-exponential (IE) distribution as a modification of the exponential distribution. Also, the IE distribution a sub-distribution of the inverted-Weibull distribution is quite a good fit for real-life processes with inverted-bathtub-failure rate characteristics. The IE distribution has a non-constant failure rate which stands good for describing real-life events

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in biomedical sciences, medicine, engineering, and public health. Abouammoh and Alshingiti (2009) introduced the generalized form of the IE distribution with the unique properties of the IE distribution. Lin et al. [2] cited in Sharifah [3], proved that the IE distribution would be efficient when used in strength-data modeling. The cumulative distribution function (CDF) of the IE distribution is given by

$$G_{IE}(y;\beta) = e^{-\frac{\beta}{y}},\ y > 0, \tag{1}$$

and the corresponding probability density function (pdf) is given by

$$g_{IE}(y;\beta) = \frac{\beta}{y^2} e^{-\frac{\beta}{y}},\ y > 0, \tag{2}$$

where $\beta > 0$ is the scale parameter.

Several authors attempting to improve the flexibility and modeling capability of the IE distribution given its quality (non-constant failure rate) and competitiveness with the exponential distribution have proposed various generalizations and extensions. Singh and Goel [4] proposed the Beta IE distribution with expressions for the inverse moments, stress-strength reliability, and inverse moment generating function. They stated that in comparison to the MLE, the Bayes estimation procedure performed better with less error value through the simulation study. Also, the real datasets applications showed that the Beta IE distribution is superior to some distributions considered using the maximum likelihood estimation (MLE) procedure. Sharifah [3] introduced the odd Fréchet IE distribution with some statistical properties and demonstrated its flexibility over some distributions using the MLE procedure. Eghwerido et al. [5] introduced the Gompertz Alpha-power IE distribution with some statistical properties and demonstrated its flexibility over some distributions using the MLE procedure on two real datasets. Leren and Abdullahi [6] proposed the odd Lindley IE distribution with decreasing hazard rate function, some statistical properties, and two real datasets considered in demonstrating the practical importance of the distribution using the MLE procedure. Abdulkadir et al. [7] proposed a two-parameter distribution called the Lomax IE distribution with derived properties such as the survival and hazard rate functions, quantile function, and order statistics. The MLE procedure utilizes fitting two sets of real datasets. Moreso, Sule [8] proposed the Topp Leone Kumaraswamy generalized IE distribution, a four-parameter distribution with an increasing-decreasing, J-and-reversed-J shaped hazard rate function. The flexibility of the distribution using the MLE was demonstrated utilizing three real datasets. Eghwerido [9] introduced the Weibull IE distribution with expressions of the moments, probability weighted moments, quantile function and then demonstrated its flexibility over some distributions using the MLE procedure.

This research aims to introduce a new extended IE distribution with unique characteristics capable of modeling skewed datasets and also examine the performance of four classical estimation procedures such as the maximum likelihood (ML), maximum product spacing (MPS), Cramer-von mises (CVM), and Anderson-Darling (ANDA) in estimating the parameters of the new extended IE distribution using extensive Monte Carlo simulation and two real datasets applications. Hence, creating a standard guideline for selecting the best estimation procedure is believed to be of interest to applied statisticians.

The remaining parts are as follows: Section 2, the exponentiated Gompertz inverted exponential (EGG_{IE}) density function, distribution function, and reliability analysis are presented. The linear representations of the EGG_{IE} functions and general structural properties are provided in Section 3. The mathematical expressions of the four estimators for the EGG_{IE} distribution are provided in Section 4. Monte Carlo simulations for the true parameter estimates of the EGG_{IE} distribution are performed using the four classical estimation procedures in Section 5. The comparison of the four estimators and the fitness ability of the EGG_{IE} distribution to two real datasets using the MLE procedure are provided in Section 6. The conclusion is given in Section 7.

2. THE EXPONENTIATED GOMPERTZ INVERTED EXPONENTIAL (EGG_{IE}) DISTRIBUTION

Alzaatreh et al. [10] proposed the method of generating classes of distributions, the CDF is given by

$$F(y) = \int_a^{H[G(y)]} v(t)dt. \tag{3}$$

By differentiating Eq (3), the corresponding pdf takes the form

$$f(y) = \left\{\frac{d}{dy} H\left[G(y)\right]\right\} v\left\{H\left[G(y)\right]\right\}. \tag{4}$$

If $H\lfloor G(y)\rfloor = -\log\left[1-G(y,\psi)\right]$ is the link function and $v(t)$ the pdf of the generalized Gompertz (GG) distribution. The CDF of the exponentiated Gompertz generated (EGG) family using Eq (3) is given by

$$F(y;\theta,\gamma,\alpha,\psi) = \int_a^{-\log\lfloor 1-G(y;\psi)\rfloor} \left[1-e^{-\frac{\theta}{\gamma}\left(e^{\gamma t}-1\right)}\right]^{\alpha} dt = \left\{1-e^{\frac{\theta}{\gamma}\left(1-\left[1-G(y;\psi)\right]^{-\gamma}\right)}\right\}^{\alpha}, \tag{5}$$

and the corresponding pdf to Eq (5) takes the form

$$f(y;\theta,\gamma,\alpha,\psi) = \frac{\alpha\theta g(y;\psi) e^{\frac{\theta}{\gamma}\left(1-\left[1-G(y;\psi)\right]^{-\gamma}\right)}}{\left[1-G(y;\psi)\right]^{1+\gamma}} \left\{1-e^{\frac{\theta}{\gamma}\left(1-\left[1-G(y;\psi)\right]^{-\gamma}\right)}\right\}^{\alpha-1}, \tag{6}$$

where $G(y;\psi)$ and $g(y;\psi)$ are the baseline CDF and pdf depending on the parameter vector ψ [11]. Therefore, the CDF of the new EGG_{IE} distribution is developed by inserting Eq (1) into Eq (5):

$$F(y;\xi) = \left\{1-e^{\frac{\theta}{\gamma}\left(1-\left[1-e^{-\frac{\beta}{y}}\right]^{-\gamma}\right)}\right\}^{\alpha}, \tag{7}$$

and the corresponding pdf to Eq (7) takes the form

$$f(y;\xi) = \frac{\alpha\theta\left(\frac{\beta}{y^2} e^{-\frac{\beta}{y}}\right) e^{\frac{\theta}{\gamma}\left(1-\left[1-e^{-\frac{\beta}{y}}\right]^{-\gamma}\right)}}{\left[1-e^{-\frac{\beta}{y}}\right]^{1+\gamma}} \left\{1-e^{\frac{\theta}{\gamma}\left(1-\left[1-e^{-\frac{\beta}{y}}\right]^{-\gamma}\right)}\right\}^{\alpha-1}, \tag{8}$$

where $\beta > 0$ is scale parameter, $\alpha, \theta, \gamma > 0$ are additional shape parameters and $y \in \Re^{+}$. From now onward, we will denote a random variable Y having pdf in Eq (8) by $Y \sim EGG_{IE}(\xi)$, where $\xi = (\alpha, \gamma, \theta, \beta)$ are the set of parameters.

Reliability analysis

The survival function (SF) of the EGG_{IE} distribution is given by

$$R_{SF}(y;\xi) = 1 - \left\{1 - e^{\frac{\theta}{\gamma}\left(1 - \left[1 - e^{-\frac{\beta}{y}}\right]^{-\gamma}\right)}\right\}^{\alpha}. \tag{9}$$

The hazard rate function (HRF) of the EGG_{IE} distribution is given by

$$h(y;\xi) = \frac{\alpha\theta \frac{\beta}{y^2} e^{-\frac{\beta}{y}} e^{\frac{\theta}{\gamma}\left(1 - \left[1 - e^{-\frac{\beta}{y}}\right]^{-\gamma}\right)}}{\left[1 - e^{-\frac{\beta}{y}}\right]^{\gamma+1}} \frac{\left\{1 - e^{\frac{\theta}{\gamma}\left(1 - \left[1 - e^{-\frac{\beta}{y}}\right]^{-\gamma}\right)}\right\}^{\alpha-1}}{\left[1 - \left\{1 - e^{\frac{\theta}{\gamma}\left(1 - \left[1 - e^{-\frac{\beta}{y}}\right]^{-\gamma}\right)}\right\}^{\alpha}\right]}. \tag{10}$$

The reversed hazard rate function (RHRF) of the EGG_{IE} distribution is given by

$$r(y;\xi) = \frac{\alpha\theta\left(\frac{\beta}{y^2} e^{-\frac{\beta}{y}}\right) e^{\frac{\theta}{\gamma}\left(1 - \left[1 - e^{-\frac{\beta}{y}}\right]^{-\gamma}\right)}}{\left[1 - e^{-\frac{\beta}{y}}\right]^{1+\gamma}} \left\{1 - e^{\frac{\theta}{\gamma}\left(1 - \left[1 - e^{-\frac{\beta}{y}}\right]^{-\gamma}\right)}\right\}^{-1}. \tag{11}$$

The odds function (OF) of the EGG_{IE} distribution is given by

$$O(y;\xi) = \frac{\left\{1 - e^{\frac{\theta}{\gamma}\left(1 - \left[1 - e^{-\frac{\beta}{y}}\right]^{-\gamma}\right)}\right\}^{\alpha}}{1 - \left\{1 - e^{\frac{\theta}{\gamma}\left(1 - \left[1 - e^{-\frac{\beta}{y}}\right]^{-\gamma}\right)}\right\}^{\alpha}}. \tag{12}$$

The cumulative hazard rate function (CHRF) of the EGG_{IE} distribution is given by

$$\mathrm{H}(y;\xi) = -\ln\left[1 - \left\{1 - e^{\frac{\theta}{\gamma}\left(1 - \left[1 - e^{-\frac{\beta}{y}}\right]^{-\gamma}\right)}\right\}^{\alpha}\right]. \tag{13}$$

3. STRUCTURAL PROPERTIES OF THE EGG$_{IE}$ DISTRIBUTION

This part inspects some fundamental structural properties of the EGG$_{IE}$ distribution. These include the series expansion of the pdf and CDF, quantile function, median, ordinary and incomplete moments, moment generating function, Lorenz and Bonferroni curves, Tsallis, Shannon, Rényi entropies, order statistics, and probability weighted moments.

3.1. Mixture representations

The mixture representations of the EGG$_{IE}$ density function (pdf) and distribution function (CDF) are derived in this subpart. According to Cordeiro et al. [11], the linear representation of Eq (5) is given by

$$F(y)=\sum_{q=0}^{\infty} b_q \mathrm{H}_q(y), \tag{14}$$

where $\mathrm{H}_q(y)$ denotes the exponential-G CDF with power parameter (q), and

$$b_q = \mathrm{I}_0(q)+\sum_{i=1}^{+\infty}\sum_{j=0}^{+\infty} \upsilon_{i,j,q}\ . \tag{15}$$

$\mathrm{I}_0(q)$ is considered as an indicator function which takes one if $q=0$, and the coefficient $\upsilon_{a,b,q}$ is given by

$$\upsilon_{a,b,q}=\frac{(-1)^{a+b+q}}{b!}\binom{\alpha}{a}\binom{-b\gamma}{q}\left(\frac{a\theta}{\gamma}\right)^{b} e^{\frac{a\theta}{\gamma}}\ . \tag{16}$$

By inserting Eq (7) into Eq (14), the CDF of the EGG$_{IE}$ distribution can be expressed as a linear combination of the IE distribution by

$$F(y)=\sum_{q=0}^{\infty} \vartheta_{a,b,q} e^{-\frac{\beta q}{y}}\ , \tag{17}$$

where $\vartheta_{a,b,q}=\sum_{a=0}^{+\infty}\sum_{b=0}^{+\infty}\frac{(-1)^{a+b+q}}{b!}\binom{\alpha}{a}\binom{-b\gamma}{q}\left(\frac{a\theta}{\gamma}\right)^{b} e^{\frac{a\theta}{\gamma}}$.

Figures 1 and 2 depict the pdf and HRF plots of the EGG$_{IE}$ distribution for selected parameter values. The pdf plots depict that the EGG$_{IE}$ distribution is unimodal, right-skew, decreasing, and increasing. In contrast, the HRF plots show an increasing and concave increasing failure rate function for the EGG$_{IE}$ distribution.

The $\left[F(y)\right]^{s}$ for the EGG$_{IE}$ distribution can be expressed as a linear combination of the inverted exponential (IE) distribution by

$$\left[F(y)\right]^{s}=\sum_{k=0}^{s} \omega_{a,b,q} e^{-\frac{\beta q}{y}}\ , \tag{18}$$

where $\omega_{a,b,q}=\sum_{a,b,q=0}^{+\infty}\frac{(-1)^{a+b+q}}{b!}\binom{s}{k}\binom{\alpha}{a}\binom{-b\gamma}{q}\left(\frac{a\theta}{\gamma}\right)^{b} e^{\frac{a\theta}{\gamma}}$ and s is an integer.

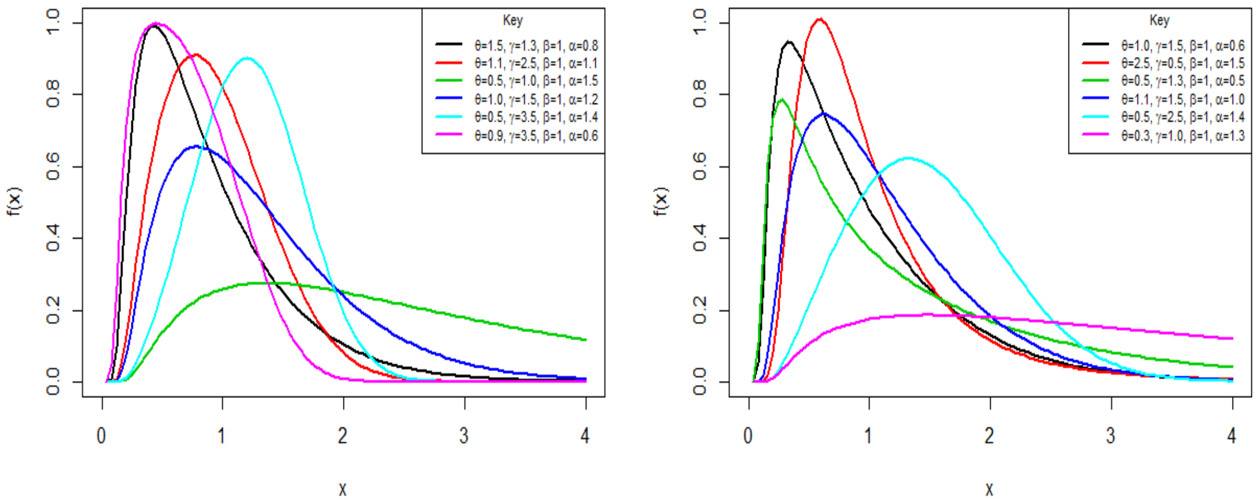

Figure 1. The EGG_{IE} pdf plots for selected values of the parameters.

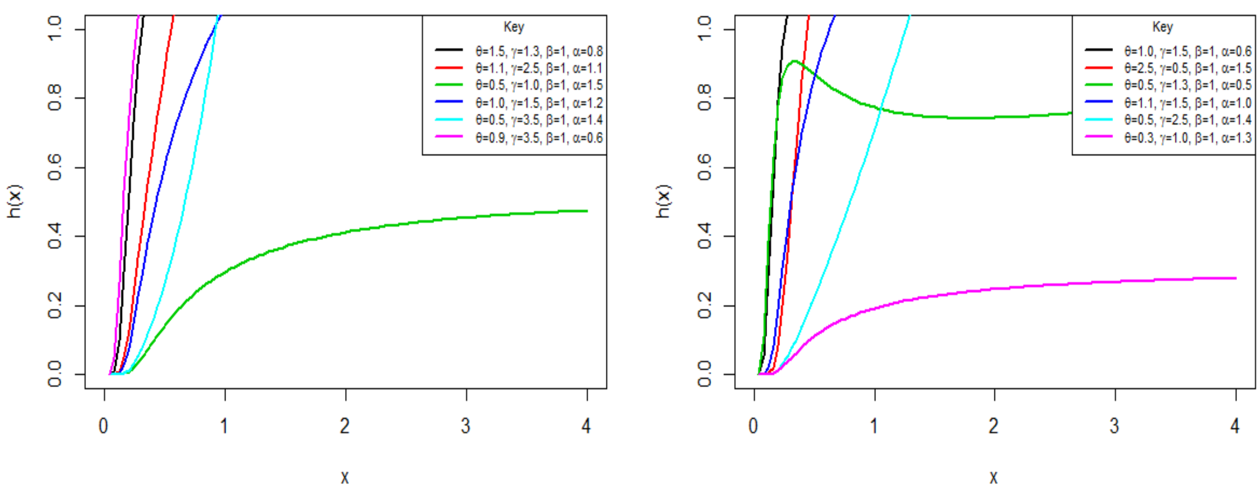

Figure 2. The EGG_{IE} HRF plots for selected values of the parameters.

According to Cordeiro et al. [11], the linear representation of the density function (pdf) of the EGG family of distributions is given by

$$f(y;\xi)=\sum_{l=0}^{\infty} b_{l+1} h_{l+1}(y;\psi), \tag{19}$$

where $h_{l+1}(y;\psi)=(l+1)g(y;\psi)G(y;\psi)^{l}$ denote the exponential-G pdf with power parameter $(l+1)$. Therefore, by inserting Eq (7) and Eq (8) into Eq (19), we have the pdf of the EGG_{IE} distribution given by

$$f(y;\xi)=\sum_{l=0}^{\infty} \frac{\Phi_{a,b,l}}{y^{2}} e^{-\frac{\beta(l+1)}{y}}, \tag{20}$$

where $\Phi_{a,b,l} = \alpha\theta\beta \sum_{a,b=0}^{+\infty} \frac{(-1)^{a+b+l}}{b!} \binom{\alpha-1}{a} \binom{-\gamma(b+1)-1}{l} \left(\frac{[(a+1)\theta]}{\gamma} \right)^b e^{\frac{(a+1)\theta}{\gamma}}$.

3.2. Quantile function

Let Y denote a random variable such that $Y \sim EGG_{IE}(\xi)$. The quantile function $Q(u),\ u \in (0,1)$ is obtained by inverting Eq (7). The quantile function of the $\mathrm{EGG_{IE}}$ distribution is given by

$$Q(u,\xi) = -\frac{\beta}{\log\left[1-\left(1-\frac{\gamma\log\left(1-u^{1/\alpha}\right)}{\theta}\right)^{-\frac{1}{\gamma}}\right]}, \qquad u \in (0,1). \tag{21}$$

By setting $u = 0.5$ in Eq (21), the median (M) function of the $\mathrm{EGG_{IE}}$ distribution is given by

$$M = -\frac{\beta}{\log\left[1-\left(1-\frac{\gamma\log\left(1-0.5^{1/\alpha}\right)}{\theta}\right)^{-\frac{1}{\gamma}}\right]}, \qquad u \in (0,1). \tag{22}$$

Most statistical software can generate uniform random variables. Hence, the quantile function Eq (21) is considered very valuable in simulating random values from the $\mathrm{EGG_{IE}}$ distribution. The quantile function Eq (21) can be utilized in estimating the skewness and kurtosis of the $\mathrm{EGG_{IE}}$ distribution. The expressions of the Bowley skewness [12] and Moor's kurtosis [13] are given by

$$S_k = \frac{Q\left(\frac{3}{4};\xi\right) - 2Q\left(\frac{1}{2};\xi\right) + Q\left(\frac{1}{4};\xi\right)}{Q\left(\frac{3}{4};\xi\right) - Q\left(\frac{1}{4};\xi\right)} \text{ and } K_s = \frac{Q\left(\frac{7}{8};\xi\right) - Q\left(\frac{5}{8};\xi\right) - Q\left(\frac{3}{8};\xi\right) + Q\left(\frac{1}{8};\xi\right)}{Q\left(\frac{6}{8};\xi\right) - Q\left(\frac{2}{8};\xi\right)} \tag{23}$$

where $Q(.)$ is the $\mathrm{EGG_{IE}}$ quantile function Eq (21).

Numerical values of the median, 25th, and 75th percentiles, skewness, and kurtosis for selected parameter values using (21), (22), and (23) are provided in Table 1. There are positively decreasing skew values and negatively increasing kurtosis values as the parameter values θ, γ and α increase, independent of the scale parameter β.

Table 1. Median (M), 25th and 75th percentiles, skewness (Sk), and kurtosis (Ks) for selected parameter values.

θ	γ	α	β	**M**	**25th**	**75th**	**Sk**	**Ks**
0.5	0.1	0.1	1.0	0.160	0.076	0.451	0.550	-2.488
1.1	0.4	0.5	1.0	0.661	0.348	1.369	0.387	-1.153
1.5	0.9	1.2	1.0	0.979	0.628	1.519	0.212	-0.570
2.5	1.8	1.9	1.0	0.808	0.609	1.053	0.104	-0.277
3.5	2.5	2.5	1.0	0.709	0.571	0.868	0.070	-0.187
4.5	3.0	3.2	1.0	0.656	0.549	0.777	0.056	-0.150
5.5	4.5	4.2	1.0	0.597	0.520	0.678	0.033	-0.088

Figures 3-5 depict the three-dimensional plots of the skewness and kurtosis measures. The plots confirm the positivity of the skewness values and the negativity of the kurtosis values.

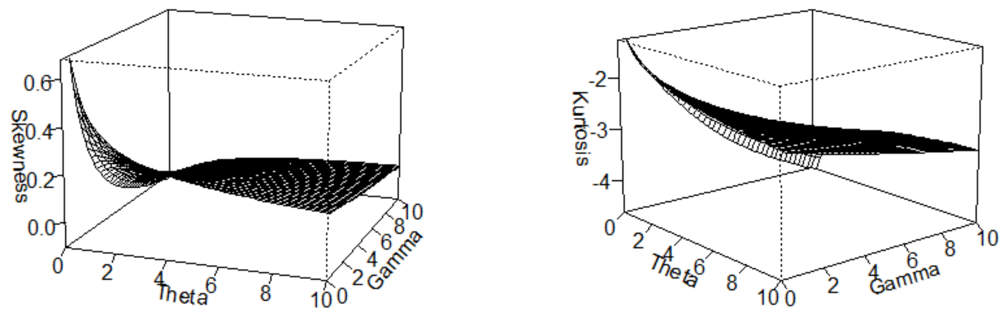

Figure 3. Bowley's skewness (left) and Moors' kurtosis (right) plots with θ = varied, γ = varied, $\alpha = 0.5$, $\beta = 1$.

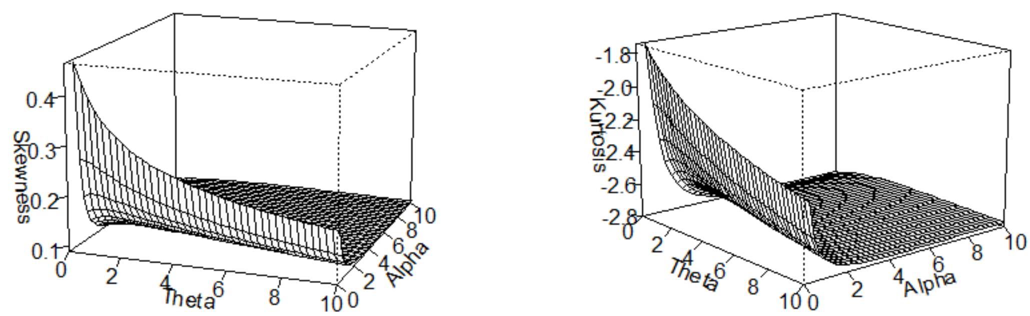

Figure 4. Bowley's skewness (left) and Moors' kurtosis (right) plots with θ = varied, γ = 1.5, α = varied, $\beta = 1$
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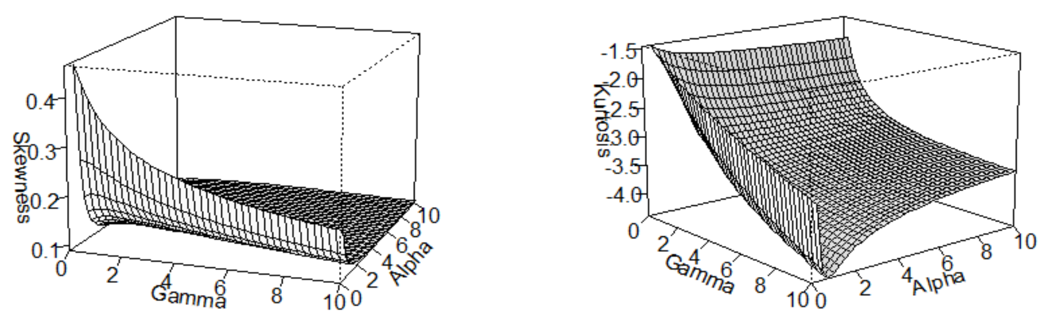

Figure 5. Bowley's skewness (left) and Moors' kurtosis (right) plots with $\theta = 0.7$, γ = vaaried, a = varied, $\beta = 1$.

3.3. Moments and moment generating function

This subpart presents the expressions of the ordinary and incomplete moments, and the moment-generating function (MGF) for the EGG_{IE} distribution.

Theorem 1. If $Y \sim EGG_{IE}(\xi)$, then the rth ordinary moment (OM) of Y is given by

$$\mu_r'(y)=\sum_{l=0}^{\infty}\Phi_{a,b,l}\frac{\Gamma(1-r)}{\left(\beta[l+1]\right)^{1-r}}, \tag{24}$$

where $\Phi_{a,b,l}=\alpha\theta\beta\sum_{a,b=0}^{+\infty}\frac{(-1)^{a+b+l}}{b!}\binom{\alpha-1}{a}\binom{-\gamma(b+1)-1}{l}\left(\frac{[(a+1)\theta]}{\gamma}\right)^{b}e^{\frac{(a+1)\theta}{\gamma}}$.

Proof. Let Y be a random variable following the EGG_{IE} distribution, the OM of Y can be derived as follows:

$$\mu_r'=\text{E}\left(Y^r\right)=\int_{-\infty}^{+\infty}y^r f(y,\xi)\,dy. \tag{25}$$

By inserting Eq (20) into Eq (25) gives

$$\mu_r'=\sum_{l=0}^{\infty}\Phi_{a,b,l}\int_0^{+\infty}y^{r-2}e^{[-\beta(l+1)]y^{-1}}dy. \tag{26}$$

Let $z=y^{-1}$, then Eq (26) takes the form

$$\mu_r'=\sum_{l=0}^{\infty}\frac{\Phi_{a,b,l}\Gamma(1-r)}{\left(\beta[l+1]\right)^{1-r}}, \tag{27}$$

where $\Phi_{a,b,l}=\alpha\theta\beta\sum_{a,b=0}^{+\infty}\frac{(-1)^{a+b+l}}{b!}\binom{\alpha-1}{a}\binom{-\gamma(b+1)-1}{l}\left(\frac{[(a+1)\theta]}{\gamma}\right)^{b}e^{\frac{(a+1)\theta}{\gamma}}$.

The moments are found by substituting $r=1,2,3,\ldots$ into Eq (27).

Theorem 2. If $Y\sim EGG_{IE}(\xi)$, then the MGF of Y is given by

$$M_Y(t)=\sum_{r,l=0}^{\infty}\frac{t^r}{r!}\Phi_{a,b,l}\frac{\Gamma(1-r)}{\left(\beta[l+1]\right)^{1-r}}. \tag{28}$$

Proof. The moment generating function is defined as

$$M_Y(t)=\text{E}\left(e^{tY}\right)=\sum_{r=0}^{\infty}\frac{t^r}{r!}\int_0^{+\infty}y^r f(y,\xi)\,dy=\sum_{r=0}^{\infty}\frac{t^r}{r!}\mu_r'(y). \tag{29}$$

By inserting Eq (27) into Eq (29) gives

$$M_Y(t)=\sum_{r,l=0}^{\infty}\frac{t^r}{r!}\frac{\Phi_{a,b,l}\Gamma(1-r)}{\left(\beta[l+1]\right)^{1-r}}, \tag{30}$$

where $\Phi_{a,b,l}=\alpha\theta\beta\sum_{a,b=0}^{+\infty}\frac{(-1)^{a+b+l}}{b!}\binom{\alpha-1}{a}\binom{-\gamma(b+1)-1}{l}\left(\frac{[(a+1)\theta]}{\gamma}\right)^{b}e^{\frac{(a+1)\theta}{\gamma}}$.

Theorem 3. If $Y\sim EGG_{IE}(\xi)$, then the s^{th} lower and upper incomplete moments (IMs) are given by

$$\varphi_s(t)=\text{E}\left(Y^s\middle|Y<t\right)=\int_0^{t}y^s f(y,\xi)\,dy, \tag{31}$$

$$\tau_s(t)=\text{E}\left(Y^s\middle|Y>t\right)=\int_t^{+\infty}y^s f(y,\xi)\,dy. \tag{32}$$

For any $s\in\text{N}$, the s^{th} lower incomplete moment (LIM) of EGG_{IE} distribution using Eq (31) is given by

$$\varphi_s(t)=\sum_{l=0}^{\infty}\Phi_{i,j,l}\int_0^t y^{s-2}e^{-\beta(l+1)y^{-1}}dy=\sum_{l=0}^{\infty}\Phi_{i,j,l}\left[\frac{v\left(1-s,\beta(l+1)t^{-1}\right)}{\left(\beta(l+1)\right)^{1-s}}\right], \tag{33}$$

where, $v(s,t)=\int_0^t y^{s-1}e^{-y}dy$ is the lower incomplete gamma function.

Likewise, the sth upper incomplete moment (UIM) of the EGG$_{IE}$ distribution using Eq (32) is given by

$$\tau_s(t)=\sum_{l=0}^{\infty}\Phi_{i,j,l}\int_t^{\infty} y^{s-2}e^{-\beta(l+1)y^{-1}}dy=\sum_{l=0}^{\infty}\Phi_{i,j,l}\left[\frac{\Gamma\left(1-s,\beta(l+1)t^{-1}\right)}{\left(\beta(l+1)\right)^{1-s}}\right], \tag{34}$$

where $\Gamma(s,t)=\int_t^{\infty} y^{s-1}e^{-y}dy$ is the upper incomplete gamma function.

The IMs are used in the calculation of other valuable statistical measures such as the mean deviation about the mean $\delta_1=\mathrm{E}(|X-\mu_1'|)$ and about the median $\delta_2=\mathrm{E}(|X-M|)$. The first LIM given as (φ_1') is useful in defining the mean deviation of X about the mean (μ_1') and median (M).

$$\delta_1=\mathrm{E}(|X-\mu_1'|)=\int_0^{+\infty}|x-\mu_1'|\,f(x;\xi)\,dx=2\mu_1'F(\mu_1')-2\varphi_1'(\mu_1'), \tag{36}$$

$$\delta_2=\mathrm{E}(|X-\mathrm{M}|)=\int_0^{+\infty}|x-\mathrm{M}|\,f(x;\xi)\,dx=\mu_1'-2\varphi_1'(\mathrm{M}),$$

where

$\mu_1'=\mu$ is the mean found by injecting $r=1$ in the OM Eq (24),

M is the median gotten by inserting $u=0.5$ in the quantile function Eq (22),

$\varphi_1'(t)=\int_0^t xf(x)dx$ is the first LIM that can be obtained by inserting $s=1$ in the LIM Eq (33).

3.4. Bonferroni and Lorenz curves

The Bonferroni and Lorenz curves are defined using the quantile function. The Bonferroni curve for the EGG$_{IE}$ distribution using the quantile function Eq (21) is given by

$$\mathrm{B}(u)=\frac{1}{u\mu}\varphi_1'\left[Q(u;\xi)\right]=\frac{1}{u\mu}\varphi_1'\left[-\frac{\beta}{\log\left[1-\left(1-\frac{\gamma\log\left(1-u^{1/\alpha}\right)}{\theta}\right)^{-\frac{1}{\gamma}}\right]}\right],\quad u\in(0,1), \tag{35}$$

and the Lorenz curve is given by

$$L(u)=\mu\mathrm{B}(u)=\frac{\mu}{u\mu}\varphi_1'\left[Q(u;\xi)\right]=\frac{1}{u}\varphi_1'\left[-\frac{\beta}{\log\left[1-\left(1-\frac{\gamma\log\left(1-u^{1/\alpha}\right)}{\theta}\right)^{-\frac{1}{\gamma}}\right]}\right],\quad u\in(0,1). \tag{36}$$

3.5. Order Statistics

Let $y_1, y_2, \ldots, y_n$ be a random sample from a continuous distribution, and $y_{1:n} < y_{2:n} < \ldots < y_{n:n}$ are order statistics (O.S) obtained from the sample. According to David [14], the z$^{\text{th}}$ O.S is given by

$$f_{z:N}(y)=\frac{g(y)}{\mathrm{B}(z,N-z+1)}\left[G(y)\right]^{z-1}\left[1-G(y)\right]^{N-z}, \tag{37}$$

where $G(y)$ and $g(y)$ are the CDF and pdf of the EGG$_{\text{IE}}$ distribution, and $\mathrm{B}(.,.)$ is the beta function. Expanding$\left[1-G(y)\right]^{N-z}$, the O.S takes the form

$$f_{z:N}(y)=\frac{1}{\mathrm{B}(z,N-z+1)}\sum_{l=0}^{N-z}(-1)^l\binom{N-z}{l}\left[G(y)\right]^{z+l-1}g(y). \tag{38}$$

By inserting Eq (7) and Eq (8) into Eq (38) and expanding the O.S equation. The expression for the O.S is given by

$$f_{z:N}(y)=\frac{1}{\mathrm{B}(z,N-z+1)}\sum_{c=0}^{\infty}\frac{\vartheta_{l,a,b,c}}{y^2}e^{-\frac{\beta(c+1)}{y}}, \tag{39}$$

where $\vartheta_{l,a,b,c}=\sum_{l=0}^{N-z}\sum_{a=o}^{\alpha(z+l)-1}\sum_{b=0}^{\infty}\frac{(-1)^{l+a+b+c}(a+1)\theta^b}{b!\gamma^b}\binom{N-z}{l}\binom{\alpha(z+l)-1}{a}\binom{-\gamma(b+1)-1}{c}e^{\frac{(a+1)\theta}{\gamma}}$.

The minimum and maximum O.S are found by setting $p=1$ and $p=n$ in Eq (39). The r$^{\text{th}}$ moment of the order statistics for the EGG$_{\text{IE}}$ distribution is given by

$$\mathrm{E}\left(X_{z:N}^r\right)=\int_{-\infty}^{+\infty}y^r f_{z:N}(y;\xi)\,dy. \tag{40}$$

By inserting Eq (39) into Eq (40), the r$^{\text{th}}$ moment of the order statistics is given by

$$\mathrm{E}\left(X_{z:N}^r\right)=\frac{1}{\mathrm{B}(z,N-z+1)}\sum_{c=0}^{\infty}\frac{\vartheta_{l,a,b,c}\Gamma(1-r)}{\left(\beta(c+1)\right)^{1-r}}, \tag{41}$$

where $\vartheta_{l,a,b,c}=\sum_{l=0}^{N-z}\sum_{a=o}^{\alpha(z+l)-1}\sum_{b=0}^{\infty}\frac{(-1)^{l+a+b+c}(a+1)\theta^b}{b!\gamma^b}\binom{N-z}{l}\binom{\alpha(z+l)-1}{a}\binom{-\gamma(b+1)-1}{c}e^{\frac{(a+1)\theta}{\gamma}}$.

3.6. Probability Weighted Moment (PWM)

The PWM of a random variable (r.v) Y is a very useful mathematical quantity in mathematical statistics [15]. The PWM of the EGG$_{\text{IE}}$ distribution is given by

$$\tau_{r,s}=\vartheta_{a,b,q,l}\frac{\Gamma(1-r)}{\left(\beta\left[q+l-1\right]\right)^{1-r}}, \tag{42}$$

where $\vartheta_{k,a,b,q,l} = \alpha\theta\beta\sum_{k=0}^{s}\sum_{a,b,q,l=0}^{+\infty}\frac{(-1)^{a+b+l+q}}{b!}\binom{s}{k}\binom{\alpha}{a}\left(\frac{a\theta+(a+1)\theta}{\gamma}\right)^{b}\binom{\alpha-1}{a}\binom{-b\gamma}{q}\binom{-\gamma(b+1)-1}{l}e^{\frac{a\theta+(a+1)\theta}{\gamma}}$.

Proof. The PWM of a r.v Y is given by

$$\tau_{r,s} = \mathrm{E}\left[Y^{r}F(y)^{s}\right] = \int_{-\infty}^{+\infty} y^{r} f(y)\left(F(y)\right)^{s} dy . \tag{43}$$

Introducing Eq (18) and Eq (20) into Eq (43) gives

$$\tau_{r,s} = \sum_{k=0}^{s}\sum_{l=0}^{\infty} a_{i,j,l}\,\omega_{i,j,q}\int_{0}^{+\infty} y^{r-2} e^{\left[-\beta(q+l+1)\right]y^{-1}} dy , \tag{44}$$

where $a_{i,j,l} = \alpha\theta\beta\sum_{i,j=0}^{+\infty}\frac{(-1)^{i+j+l}}{j!}\binom{\alpha-1}{i}\binom{-\gamma(j+1)-1}{l}\left(\frac{\left[(i+1)\theta\right]}{\gamma}\right)^{j} e^{\frac{(i+1)\theta}{\gamma}}$,

$$\omega_{i,j,q} = \sum_{i,j,q=0}^{+\infty}\frac{(-1)^{i+j+q}}{j!}\binom{s}{k}\binom{\alpha}{i}\binom{-j\gamma}{q}\left(\frac{i\theta}{\gamma}\right)^{j} e^{\frac{i\theta}{\gamma}} .$$

Let $z = y^{-1}$, then

$$\tau_{r,s} = \vartheta_{i,j,q,l}\int_{0}^{+\infty} z^{-r} e^{\left[-\beta(q+l+1)\right]z} dz , \tag{45}$$

where $\vartheta_{i,j,q,l} = a_{i,j,l}\,\omega_{i,j,q}$.

Therefore, the PWM of the $\mathrm{EGG_{IE}}$ distribution is given by

$$\tau_{r,s} = \vartheta_{i,j,q,l}\frac{\Gamma(1-r)}{\left(\beta\left[q+l+1\right]\right)^{1-r}} , \tag{46}$$

where $\vartheta_{i,j,q,l} = \alpha\theta\beta\sum_{k=0}^{s}\sum_{i,j,q,l=0}^{+\infty}\frac{(-1)^{i+j+l+q}}{j!}\binom{s}{k}\binom{\alpha}{i}\left(\frac{i\theta+(i+1)\theta}{\gamma}\right)^{j}\binom{\alpha-1}{i}\binom{-j\gamma}{q}\binom{-\gamma(j+1)-1}{l}e^{\frac{i\theta+(i+1)\theta}{\gamma}}$.

3.7. Entropy

This subpart, the most popular entropy (EPY) measures known as the Shannon, Tsallis and Rényi entropies ([16] [17] [18]) are derived. The Shannon EPY is given by

$$S_{E} = \mathrm{E}\left\{-\log\left[f(y)\right]\right\} . \tag{47}$$

The Shannon EPY for the $\mathrm{EGG_{IE}}$ distribution is given by

$$\begin{aligned}\mathrm{E}\left[-\log\left(f(y)\right)\right] &= -\log(\alpha\theta) - \mathrm{E}\left\{\log\left(\frac{\beta}{y^{2}}e^{-\beta/x}\right)\right\} + \frac{(\gamma+1)\alpha\theta}{\gamma^{2}}\sum_{i,j=0}^{\infty}\frac{(-1)^{i+j+1}}{(j+1)(j+1)!}\binom{\alpha-1}{i}\left[\frac{(i+1)\theta}{\gamma}\right]^{j} e^{\frac{(i+1)\theta}{\gamma}} \\ &-\frac{\theta}{\gamma}\left\{1-\frac{\alpha\theta}{\gamma^{2}}\sum_{i,j=0}^{\infty}\frac{(-1)^{i+j}}{(j+2)\,j!}\binom{\alpha-1}{i}\left[\frac{(i+1)\theta}{\gamma}\right]^{j}\right\}e^{\frac{(i+1)\theta}{\gamma}} - \frac{\alpha(\alpha+1)\theta}{\gamma}\sum_{i,j=0}^{\infty}\frac{(-1)^{i+j+1}}{(j+1)!}\left[\frac{(i+1)\theta}{\gamma}\right]^{j} e^{\frac{(i+1)\theta}{\gamma}} \\ &\left[\frac{\partial}{\partial t}\binom{t+\alpha+1}{i}\Bigg|_{t=0}\right].\end{aligned} \tag{48}$$

The Rényi EPY is given by

$$\mathrm{I}_{R(\delta)} = (1-\delta)^{-1}\log\int_{-\infty}^{+\infty} f(y)^{\delta} dy , \text{ where } \delta > 0 \text{ and } \delta \neq 1 .$$

The Rényi for the $\mathrm{EGG_{IE}}$ is derived as follows:

$$\mathrm{I}_{R(\delta)} = (1-\delta)^{-1}\log\left[\eta_{i,j,k}\int_{0}^{+\infty} y^{-2\delta} e^{\left[-\beta(k+\delta)\right]y^{-1}} dy\right] , \tag{49}$$

where $\eta_{i,j,k}=\sum_{i,j,k=0}^{\infty}\alpha^{\delta}\theta^{\delta}\beta^{\delta}\frac{(-1)^{i+j+k}\{(i+\delta)\theta\}^{j}}{j!\gamma^{j}}\binom{\delta(\alpha-1)}{i}\binom{-\delta(\gamma+1)-\gamma j}{k}e^{\frac{\theta(i+\delta)}{\gamma}}$.

Let $z=y^{-1}$, then

$$\eta_{i,j,k}\int_{0}^{+\infty}z^{-\delta}e^{[-\beta(k+\delta)]z}dz=\eta_{i,j,k}\frac{\Gamma(1-2\delta)}{[\beta(k+\delta)]^{2\delta-1}}. \quad (50)$$

Thus, the Rényi EPY for the EGG$_{\mathrm{IE}}$ distribution is given by

$$\mathrm{I}_{R(\delta)}=(1-\delta)^{-1}\log\left[\eta_{i,j,k}\frac{\Gamma(1-2\delta)}{[\beta(k+\delta)]^{2\delta-1}}\right]. \quad (51)$$

Likewise, the Tsallis EPY for the EGG$_{\mathrm{IE}}$ distribution using Eq (51) is given by

$$\mathrm{H}_{q}=(q-1)^{-1}\log\left[1-\left(\eta_{i,j,k}\frac{\Gamma(1-2q)}{[\beta(k+q)]^{2q-1}}\right)\right]. \quad (52)$$

4. ESTIMATION METHODS

This part discusses four classical estimation procedures for estimating the parameters of the EGG$_{\mathrm{IE}}$ distribution. These estimators are the maximum likelihood (ML), maximum product-spacing (MPS), Anderson-Darling (ANDA), and Cramer-von mises (CVM) procedures.

4.1. The ML

Let $y_1,y_2,\ldots,y_n$ be the sample values from the EGG$_{\mathrm{IE}}$ distribution with an unknown parameter vector $\xi=(\theta,\alpha,\gamma,\beta)^{T}$. The log-likelihood function (l) of the EGG$_{\mathrm{IE}}$ density function (pdf) is given by

$$\begin{aligned}l=\log L(\xi)&=n\log\alpha+n\log\theta+\sum_{i=1}^{n}\log\left(\frac{\beta}{x_i^{2}}e^{-\frac{\beta}{x_i}}\right)-(\gamma+1)\sum_{i=1}^{n}\log(c_i)\\&+\sum_{i=1}^{\infty}\log(1-b_i)+(\alpha-1)\sum_{i=1}^{n}\log(b_i)\end{aligned} \quad (53)$$

where $b_i=1-e^{\frac{\theta}{\gamma}\left(1-(1-e^{-\beta/x_i})^{-\gamma}\right)}$ and $c_i=\left(1-e^{-\beta/x_i}\right)$. The associated score function is provided by $U(\xi)=\left[\frac{\partial l}{\partial\theta},\frac{\partial l}{\partial\gamma},\frac{\partial l}{\partial\alpha},\frac{\partial l}{\partial\beta}\right]^{T}$. Score function components gotten by differentiating the nonlinear Eq (53) are given by

$$\frac{\partial l}{\partial\theta}=\frac{n}{\theta}+\frac{1}{\gamma}\sum_{i=1}^{n}\left(1-(c_i)^{-\gamma}\right)-\frac{(\alpha-1)}{\gamma}\sum_{i=1}^{n}\frac{\left(1-(c_i)^{-\gamma}\right)e^{\frac{\theta}{\gamma}\{1-[c_i]^{-\gamma}\}}}{b_i}, \quad (54)$$

$$\frac{\partial l}{\partial\gamma}=-\sum_{i=1}^{n}\log(c_i)-\sum_{i=1}^{n}\frac{b_i^{(\gamma)}}{1-b_i}+(\alpha-1)\sum_{i=1}^{n}\frac{b_i^{(\gamma)}}{b_i}, \quad (55)$$

$$\frac{\partial l}{\partial\alpha}=\frac{n}{\alpha}+\sum_{i=1}^{n}\log(b_i), \quad (56)$$

and

$$\frac{\partial l}{\partial \beta}=\frac{1}{\beta}\sum_{i=1}^{n}\frac{(x_i-\beta)}{x_i}-(\gamma+1)\sum_{i=1}^{n}\frac{e^{-\frac{\beta}{x_i}}}{x_i(c_i)}+\theta\sum_{i=1}^{n}\frac{e^{-\frac{\beta}{x_i}}(c_i)^{-\gamma-1}}{x_i} \\ -\theta(\alpha-1)\sum_{i=1}^{n}\frac{e^{-\frac{\beta}{x_i}}(c_i)^{-\gamma}e^{\frac{\theta}{\gamma}\{1-[c_i]^{-\gamma}\}}}{x_i(c_i)(b_i)}. \tag{57}$$

4.2. The MPS

The MPS proposed by Cheng and Amin [19] [20] and developed by Ranneby [21], is a good substitute to the ML. Let $y_{(1:n)},y_{(2:n)},\ldots,y_{(n:n)}$ be the ordered sample from the EGG$_{IE}$ distribution with parameter vector $\xi=(\theta,\alpha,\gamma,\beta)$ and the uniform spacing for this random sample. The expression of the MPS is given by

$$D_i(\xi)=F\left(y_{(i)}|\xi\right)-F\left(y_{(i-1)}|\xi\right),\text{ for } i=1,2,\ldots,n+1,$$

where $F\left(y_{(0)}|\xi\right)=0$, $F\left(y_{(n+1)}|\xi\right)=1$, such that $\sum_{i=1}^{n+1}D_i(\xi)=1$. Then,

$$F\left(y_{(i)}|\xi\right)=\left\{1-e^{\frac{\theta}{\gamma}\left(1-\left[1-e^{-\frac{\beta}{x_{(i)}}}\right]^{-\gamma}\right)}\right\}^{\alpha}\text{ and } F\left(y_{(i-1)}|\xi\right)=\left\{1-e^{\frac{\theta}{\gamma}\left(1-\left[1-e^{-\frac{\beta}{x_{(i-1)}}}\right]^{-\gamma}\right)}\right\}^{\alpha}.$$

The MPS estimates of $\hat{\theta}_{MPS}$, $\hat{\gamma}_{MPS}$, $\hat{\alpha}_{MPS}$ and $\hat{\beta}_{MPS}$ are found by maximising with respect to θ,γ,α and β.

$$G(\xi)=\left\langle\prod_{i=1}^{n+1}D_i(\xi)\right\rangle^{\frac{1}{n+1}}\quad\text{and}\quad H(\xi)=\frac{1}{n+1}\sum_{i=1}^{n+1}\log\left\langle D_i(\xi)\right\rangle$$

The EGG$_{IE}$ parameter estimates using MPS can be found by solving the following nonlinear equations

$$\frac{1}{n+1}\sum_{i=1}^{n+1}\frac{1}{D_i(\xi)}\left\langle\Delta_t\left(y_{(i)}|\xi\right)-\Delta_t\left(y_{(i-1)}|\xi\right)\right\rangle=0,\text{ for } t=1,2,3,4.$$

where

$$\Delta_1\left(y_{(i)}|\xi\right)=\frac{\partial}{\partial\theta}F\left(y_{(i)}|\xi\right), \\ \Delta_2\left(y_{(i)}|\xi\right)=\frac{\partial}{\partial\gamma}F\left(y_{(i)}|\xi\right), \\ \Delta_3\left(y_{(i)}|\xi\right)=\frac{\partial}{\partial\alpha}F\left(y_{(i)}|\xi\right), \\ \Delta_4\left(y_{(i)}|\xi\right)=\frac{\partial}{\partial\beta}F\left(y_{(i)}|\xi\right). \tag{58}$$

It is important to state that Δ_t, for $t=1,2,3,4$ can be obtained numerically.

4.3. The ANDA

The ANDA [22] estimates of the EGG$_{IE}$ distribution with parameters vector $\xi=(\theta,\alpha,\gamma,\beta)$ can be found by minimizing the function

$$AD(\xi) = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left\langle \log\left[F\left(y_{(i)}|\xi\right)\right] + \log\left[1 - F\left(y_{(n+1-i)}|\xi\right)\right]\right\rangle .$$

Solving the following nonlinear equations, the ANDA estimates $\hat{\theta}_{ANDA}$, $\hat{\gamma}_{ANDA}$, $\hat{\alpha}_{ANDA}$ and $\hat{\beta}_{ANDA}$ can also be found using

$$\sum_{i=1}^{n}(2i-1)\left\langle \frac{\Delta_t\left(y_{(i)}|\xi\right)}{F\left(y_{(i)}|\xi\right)} - \frac{\Delta_k\left(y_{(n+1-i)}|\xi\right)}{1 - F\left(y_{(n+1-i)}|\xi\right)}\right\rangle = 0, \text{ for } t,k = 1,2,3,4.$$

where $\Delta_1(\cdot|\xi)$, $\Delta_2(\cdot|\xi)$, $\Delta_3(\cdot|\xi)$ and $\Delta_4(\cdot|\xi)$ are given in (58).

4.4. The CVM

The CVM ([22] [23]) estimates of the EGG_{IE} distribution with parameters vector $\xi = (\theta, \alpha, \gamma, \beta)$ can be found by minimizing function

$$C(\xi) = \frac{1}{12n} + \sum_{i=1}^{n}\left\langle \left\{\frac{2i-1}{2n}\right\} - F\left(y_{(i)}|\xi\right)\right\rangle^2 ,$$

Solving the following nonlinear equations, the CVM estimates $\hat{\theta}_{CVM}$, $\hat{\gamma}_{CVM}$, $\hat{\alpha}_{CVM}$ and $\hat{\beta}_{CVM}$ can also be found using

$$\sum_{i=1}^{n}\left\langle F\left(y_{(i)}|\xi\right) - \left\{\frac{2i-1}{2n}\right\}\right\rangle \Delta_t\left(y_{(i)}|\xi\right) = 0, \text{ for } t = 1,2,3,4.$$

where $\Delta_1(\cdot|\xi)$, $\Delta_2(\cdot|\xi)$, $\Delta_3(\cdot|\xi)$ and $\Delta_4(\cdot|\xi)$ are given in (58).

5. SIMULATION STUDY

The ML, MPS, ANDA, and CVM for the EGG_{IE} distribution are evaluated using Monte Carlo simulations. The performance of the procedures was evaluated using the average estimates (AEs), Root Mean Square Errors (RMSE), absolute biases (ABS), and Mean Square Errors (MSE) for different sample sizes. $N = 3000$ Samples are generated from the EGG_{IE} distribution, each sample size $n = 20, 50, 150, 300, 1000$ for selected parameter values $\theta = 1.0$, $\gamma = 1.2$, $\beta = 1.0$ and $\alpha = 2.5$. These parameter values are arbitrarily chosen to assess the procedures' ability to estimate the parameters of the EGG_{IE} distribution with a minimum bias for small and large data samples. The ABS, MSE, and RMSE are computed for $\hat{S} = \hat{\theta}, \hat{\gamma}, \hat{\alpha}, \hat{\beta}$ using

$$Abs\hat{B}ias_s = \frac{1}{N}\sum_{i=1}^{N}\left|\hat{S}_i - S\right|, \ \hat{M}SE_s = \frac{1}{N}\sum_{i=1}^{N}\left(\hat{S}_i - S\right)^2, \ \hat{R}MSE_s = \sqrt{\frac{1}{N}\sum_{i=1}^{N}\left(\hat{S}_i - S\right)^2} .$$

The simulation results are provided in Table A1 (Appendix A). The results show that the MSE and RMSE values decrease as the sample size increase for all the procedures.

6. APPLICATION

The flexibility and superiority of the EGG_{IE} distribution in relation to some existing competing distributions are demonstrated using two real datasets applications. The first dataset consists of 63 observations of the strengths of 1.5 cm glass fibers obtained by employees at the UK National Physical Laboratory. The observations are as follows:

0.55, 0.74, 0.77, 0.81, 0.84, 1.24, 0.93, 1.04, 1.11, 1.13, 1.30, 1.25, 1.27, 1.28,1.29, 1.48, 1.36, 1.39, 1.42, 1.48, 1.51, 1.49, 1.49, 1.50, 1.50,1.55, 1.52, 1.53, 1.54, 1.55, 1.61, 1.58, 1.59, 1.60, 1.61, 1.63,1.61, 1.61, 1.62, 1.62, 1.67, 1.64, 1.66, 1.66, 1.66, 1.70, 1.68,1.68, 1.69, 1.70, 1.78, 1.73, 1.76, 1.76, 1.77, 1.89, 1.81, 1.82,1.84, 1.84, 2.00, 2.01, 2.24. Recently, the dataset had been studied by Abouelmagd et al. [24], Mead et al. [25], Zelibe et al. [26] and Eghwerido [9].

The second dataset consists of the breaking stress of carbon fibers of 50 mm length (GPa). The observations are as follows:
0.39, 0.85, 1.08, 1.25, 1.47, 1.57, 1.61, 1.61, 1.69, 1.80, 1.84, 1.87, 1.89, 2.03, 2.03, 2.05, 2.12, 2.35, 2.41, 2.43, 2.48, 2.50, 2.53, 2.55, 2.55, 2.56, 2.59, 2.67, 2.73, 2.74, 2.79, 2.81, 2.82, 2.85, 2.87, 2.88, 2.93, 2.95, 2.96, 2.97, 3.09, 3.11, 3.11, 3.15, 3.15, 3.19, 3.22, 3.22, 3.27, 3.28, 3.31, 3.31, 3.33, 3.39, 3.39, 3.56, 3.60, 3.65, 3.68, 3.70, 3.75, 4.20, 4.38, 4.42, 4.70, 4.90. The dataset was recently studied by AL-Bastian et al. [27].

The EGG_{IE} distribution parameters are estimated using the maximum likelihood estimation procedure. We compare the EGG_{IE} distribution with the following competing distributions such as the Gompertz Lomax (GOLOM) by Oguntunde et al. [28], Odd Fréchet Inverse exponential (OFIE) by Sahrifah [3], exponentiated generalized inverse exponential (EGIE) by Oguntunde et al. [29], generalized inverse exponential (GIE) by Abouammoh and Alshingiti [30], exponential inverse exponential (EIE) by Oguntunde et al. [31], Gompertz Weibull (GOWE) by [32]. The following performance measures such as the Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), Consistent Akaike Information Criterion (CAIC), Hannan-Quinn information criterion (HQIC), negative log likelihood (-LL), Anderson Darling statistic (ANDA), Cramer-von Mises statistic (CVM), and Kolmogorov-Smirnov test (KS) and its p-value are computed using the R-environment (*AdequacyModel package*). The distribution with the least performance measure values produces the best goodness of fit for the datasets.

The descriptive statistics of the datasets are provided in Tables 2 and 3. From the tables, it is observed that both datasets are left skewed and leptokurtic.

Table 2. Descriptive statistics (first dataset).

Mean	Median	Min	Max	1st Qu	3rd Qu	skewness	Kurtosis
1.507	1.590	0.550	2.240	1.375	1.685	-0.899	0.924

Table 3. Descriptive statistics (second dataset).

Mean	Median	Min	Max	1st Qu	3rd Qu	skewness	Kurtosis
2.760	2.835	0.390	4.900	2.178	3.277	-0.131	0.223

The box plots in Figures 7_a and 9_a show that the first and second datasets are left skewed. The total time on test (TTT) plots in Figures 7_b and 9_b depict concave increasing failure rates for the first and second datasets. The ML parameter estimates and the standard errors (SEs) of the distributions using the first and second datasets are provided in Tables 4 and 5.

Table 4. The first dataset: MLEs and SE (in parentheses).

Model	MLE and SE			
$EGG_{IE}(\alpha,\beta,\theta,\gamma)$	1.055 (0.337)	2.023 (1.371)	0.243 (0.685)	11.304 (5.436)
$EIE(\alpha,\beta)$	1.187 (91.427)	1.187 (91.427)	-	-
$GIE(\alpha,\beta)$	106.381 (44.324)	7.486 (0.689)	-	-
$EGIE(\alpha,\beta,\theta)$	144.004 (106.243)	8.177 (1.502)	0.821 (0.233)	-
$OFIE(\alpha,\beta)$	1.038 (0.086)	0.894 (0.115)	-	-
$GOLO(\alpha,\beta,\theta,\gamma)$	1.516 (0.450)	0.507 (0.153)	0.005 (0.002)	8,179 (2.298)
$GOWE(\alpha,\beta,\theta,\gamma)$	0.798 (0.514)	5.615 (0.510)	0.223(0.812)	0.009 (0.046)

Table 5. The second dataset: MLEs and SE (in parentheses).

Model	MLE and SE			
$EGG_{IE}(\alpha,\beta,\theta,\gamma)$	0.999 (0.365)	1.041 (0.678)	0.022 (0.037)	4.182 (1.222)
$EIE(\alpha,\beta)$	1.516 (145.850)	1.516 (145.850)	-	-
$GIE(\alpha,\beta)$	13.279 (4.264)	7.600 (0.904)	-	-
$EGIE(\alpha,\beta,\theta)$	36.688 (0.100)	12.914 (0.016)	0.444 (0.055)	-
$OFIE(\alpha,\beta)$	1.129 (0.085)	0.703 (0.087)	-	-
$GOLO(\alpha,\beta,\theta,\gamma)$	0.605 (1.160)	0.898 (0.812)	0.013 (0.013)	8.002 (13.942)
$GOWE(\alpha,\beta,\theta,\gamma)$	0.321 (0.122)	3.399 (0.577)	1.048 (0.546)	0.020 (0.008)

The measures used in evaluating the fitness performance of the distributions on the two datasets are provided in Tables 6 and 7. The empirical results in Tables 6 and 7 show that the EGG_{IE} distribution has the lowest measure values for the two datasets, implying that the EGG_{IE} distribution provides a better fit to the two datasets than the other competing distributions previously mentioned.

Table 6. The first dataset: Performance measures.

Model	AIC	CAIC	BIC	HQIC	ANDA	CVM	-LL	KS	p-value
$EGG_{IE}(\alpha,\beta,\theta,\gamma)$	36.054	36.744	44.627	39.426	0.946	0.169	14.027	0.133	0.214
$EIE(\alpha,\beta)$	182.878	183.078	187.165	184.564	4.666	0.860	89.439	0.488	1.886e-13
$GIE(\alpha,\beta)$	48.745	48.965	53.051	50.451	2.812	0.514	22.382	0.207	0.009
$EGIE(\alpha,\beta,\theta)$	48.680	49.087	55.111	51.209	2.622	0.480	21.340	0.221	0.004
$OFIE(\alpha,\beta)$	148.767	148.967	153.054	150.453	6.254	1.179	72.384	0.438	6.468e-11
$GOLO(\alpha,\beta,\theta,\gamma)$	37.005	37.695	45.578	40.377	0.946	0.168	14.502	0.154	0.100
$GOWE(\alpha,\beta,\theta,\gamma)$	38.377	39.066	46.949	41.748	1.283	0.233	15.188	0.152	0.109

Figures 6 and 8 depict the fitted density function (pdf) plot, distribution function (CDF) plot, probability-probability (PP) plot, and quantile-quantile (QQ) plot of the EGG_{IE} distribution for the two datasets. The plots support the results presented in Tables 6 and 7, the EGG_{IE} distribution provides the best goodness of fit to the two datasets. The hazard rate and survival function plots in

Figures $7_{(c,\ d)}$ and $8_{(c,\ d)}$, reveal that the EGG_{IE} distribution is very relevant in reliability and survival studies. Additionally, the confidence interval of the parameter estimates for the EGG_{IE} distribution provided in Table 8 indicates that the estimated parameter values are within the confidence bounds.

Table 7. The second dataset: Performance measures.

Model	AIC	CAIC	BIC	HQIC	ANDA	CVM	-LL	KS	p-value
$EGG_{IE}(\alpha,\beta,\theta,\gamma)$	179.177	179.833	187.935	182.638	0.464	0.076	85.588	0.081	0.775
$EIE(\alpha,\beta)$	276.057	276.247	280.436	277.787	3.863	0.684	136.028	0.383	8.124e-9
$GIE(\alpha,\beta)$	203.240	203.431	207.620	204.971	2.384	0.425	99.620	0.167	0.050
$EGIE(\alpha,\beta,\theta)$	195.111	195.499	201.681	197.708	1.689	0.301	94.555	0.169	0.046
$OFIE(\alpha,\beta)$	277.059	277.249	281.438	278.789	7.979	1.448	136.529	0.405	7.643e-10
$GOLO(\alpha,\beta,\theta,\gamma)$	179.345	180.001	188.103	182.806	0.441	0.068	85.672	0.085	0.722
$GOWE(\alpha,\beta,\theta,\gamma)$	180.127	180.783	188.886	183.588	0.528	0.093	86.064	0.082	0.761

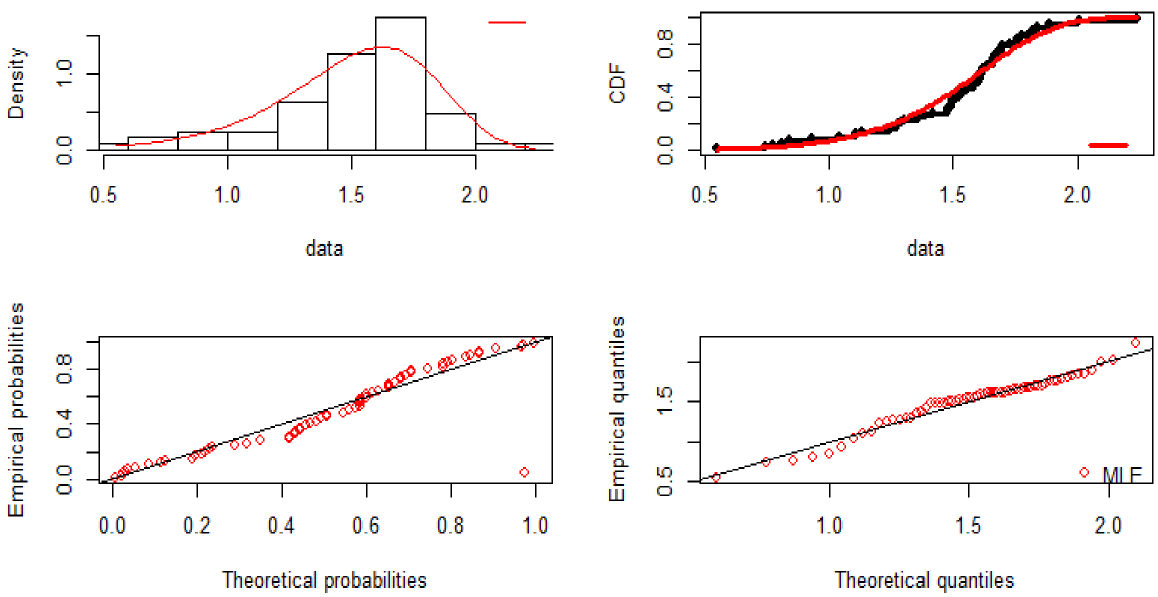

Figure 6. Fitted density function (pdf) plot (top left panel), Fitted distribution function (CDF) plot (top right panel), Fitted PP plot (bottom left panel) and Fitted QQ plot (bottom right panel) for the EGG_{IE} distribution using the first dataset.

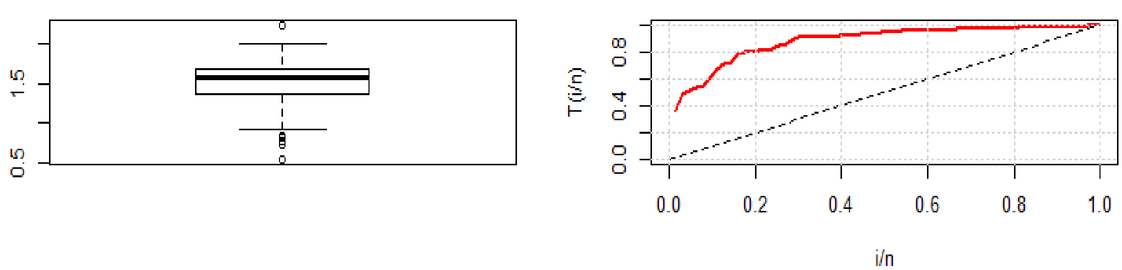

Figure 7. *Cont.*

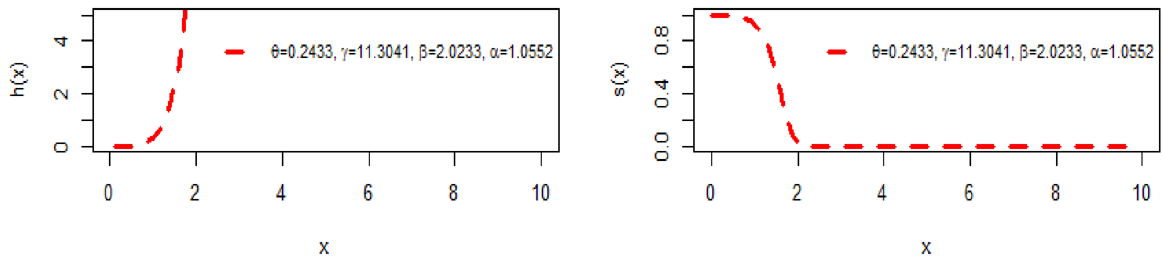

Figure 7. Box plot (top left panel), TTT plot (top right panel), fitted hazard rate function plot (bottom left panel), and fitted survival function plot (bottom right panel) for the EGG$_{IE}$ distribution using the first dataset.

Table 8. Parameter estimates confidence intervals for the EGG$_{IE}$ distribution.

CI	$\hat{\theta}$	$\hat{\gamma}$	$\hat{\alpha}$	$\hat{\beta}$
First dataset				
95%	$[-1.0996 \quad 1.5856]$	$[0.6494 \quad 21.9586]$	$[0.3945 \quad 1.7155]$	$[-0.6549 \quad 4.7195]$
99%	$[-1.5174 \quad 2.0034]$	$[-2.6665 \quad 25.2745]$	$[0.1889 \quad 1.9211]$	$[-1.4912 \quad 5.5558]$
Second dataset				
95%	$[-0.0505 \quad 0.0945]$	$[1.7869 \quad 6.5771]$	$[0.2836 \quad 1.7144]$	$[-0.2879 \quad 2.3699]$
99%	$[-0.0731 \quad 0.1171]$	$[1.0415 \quad 7.3225]$	$[0.0609 \quad 1.9370]$	$[-0.7015 \quad 2.7835]$

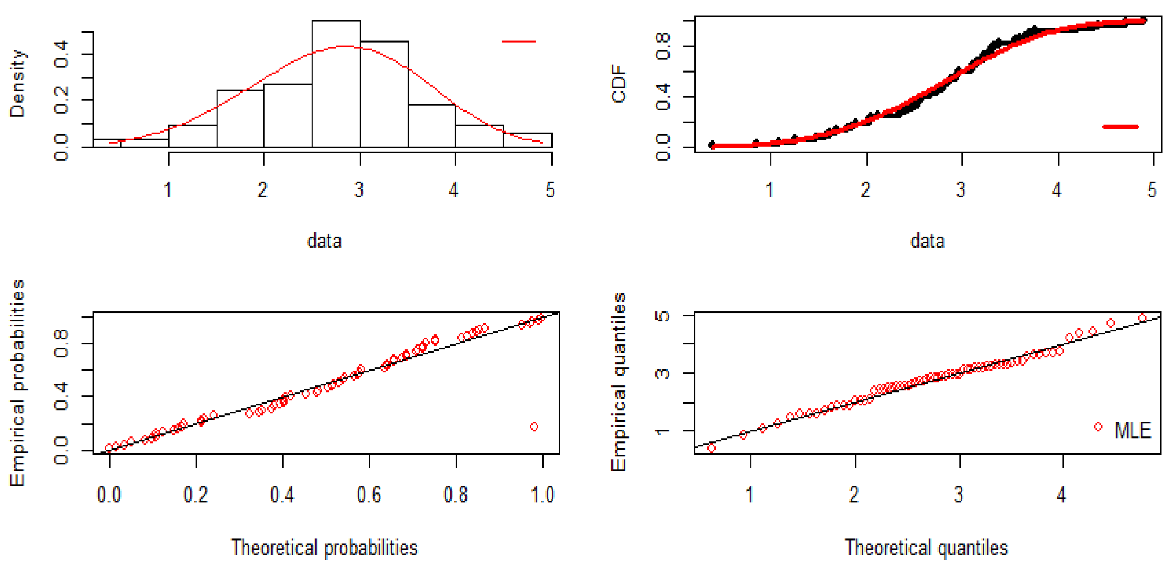

Figure 8. Fitted density function (pdf) plot (top left panel), Fitted distribution function (CDF) plot (top right panel), Fitted PP plot (bottom left panel) and Fitted QQ plot (bottom right panel) for the EGG$_{IE}$ distribution using the second dataset.

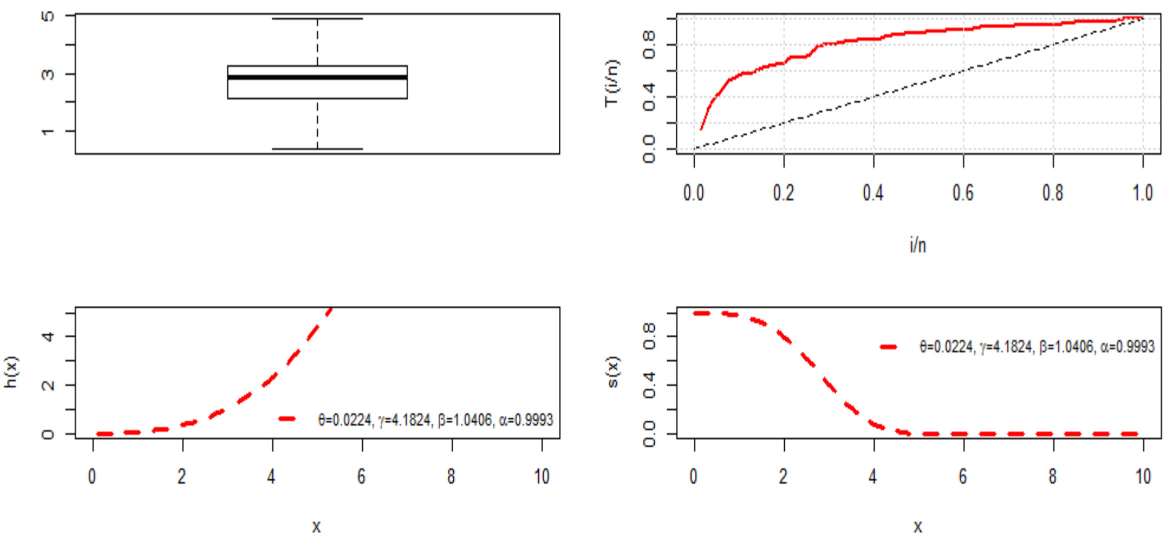

Figure 9. Box plot (top left panel), TTT plot (top right panel), fitted hazard rate function plot (bottom left panel), and fitted survival function plot (bottom right panel) for the EGG_{IE} distribution using the second dataset.

The EGG_{IE} is fitted to the two real datasets using MLE, MPSE, CVME, and ANDE procedures. Tables 9 and 10 provide the estimated parameter values, the K.S test statistic, and the p-value for the K.S statistic. From Tables 9 and 10, the K.S test statistic and p-values show that the CVME is the best, followed by ANDE among all the procedures considered. The histograms and fitted pdfs for the first and second datasets are presented in Figures 10 and 11. The plots confirm the CVM procedure provides the best-estimated parameter values for the first and second datasets.

Table 9. The first dataset: Parameter estimates of the EGG_{IE} distribution with the four procedures.

Model	Parameter estimates				Goodness of fit	
	$\hat{\theta}$	$\hat{\gamma}$	$\hat{\alpha}$	$\hat{\beta}$	K. S	p-value
CVME	0.0414	12.9362	1.0413	1.6796	0.081	0.804
ANDE	0.0457	15.2839	0.7649	1.9488	0.107	0.465
MLE	0.2433	11.3041	1.0552	2.0233	0.133	0.214
MPSE	0.2464	10.4682	0.9932	1.9726	0.144	0.148

Table 10. The second dataset: Parameter estimates of the EGG_{IE} distribution with the four procedures.

Model	Parameter estimates				Goodness of fit	
	$\hat{\theta}$	$\hat{\gamma}$	$\hat{\alpha}$	$\hat{\beta}$	K. S	p-value
CVME	0.3211	11.9372	0.6666	4.3901	0.070	0.905
ANDE	0.0512	4.4329	1.0824	1.3731	0.072	0.879
MLE	0.0224	4.1824	0.9993	1.0406	0.081	0.775
MPSE	0.0128	3.9358	0.9313	0.8354	0.092	0.628

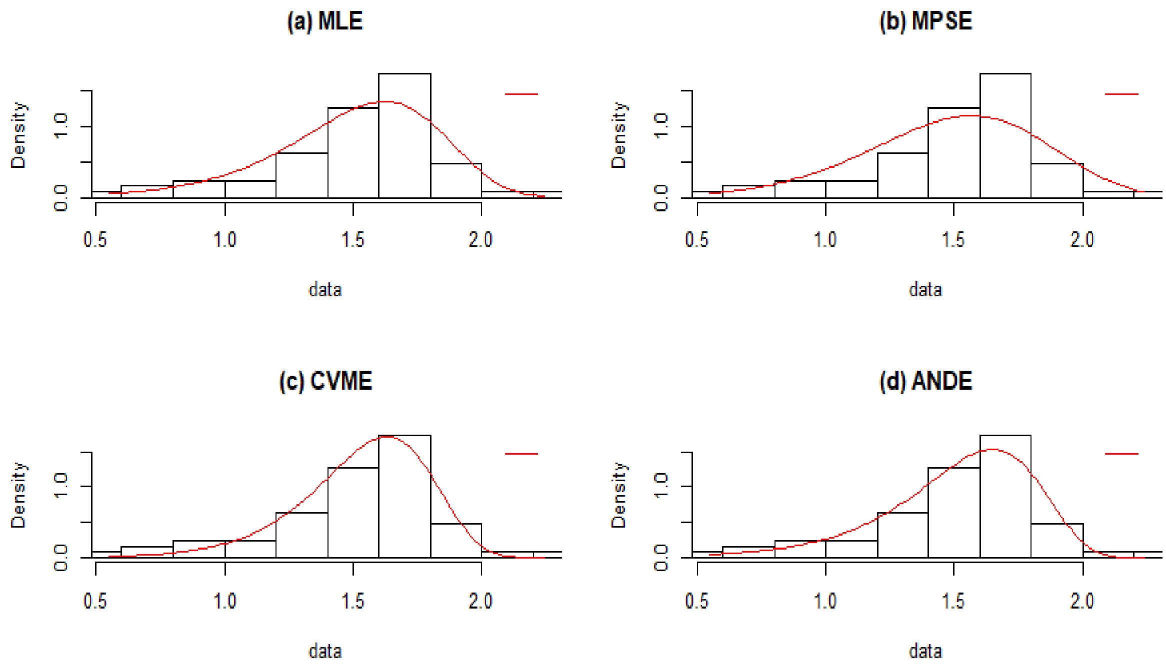

Figure 10. The EGG_{IE} fitted density function on the first dataset histogram with the four procedures.

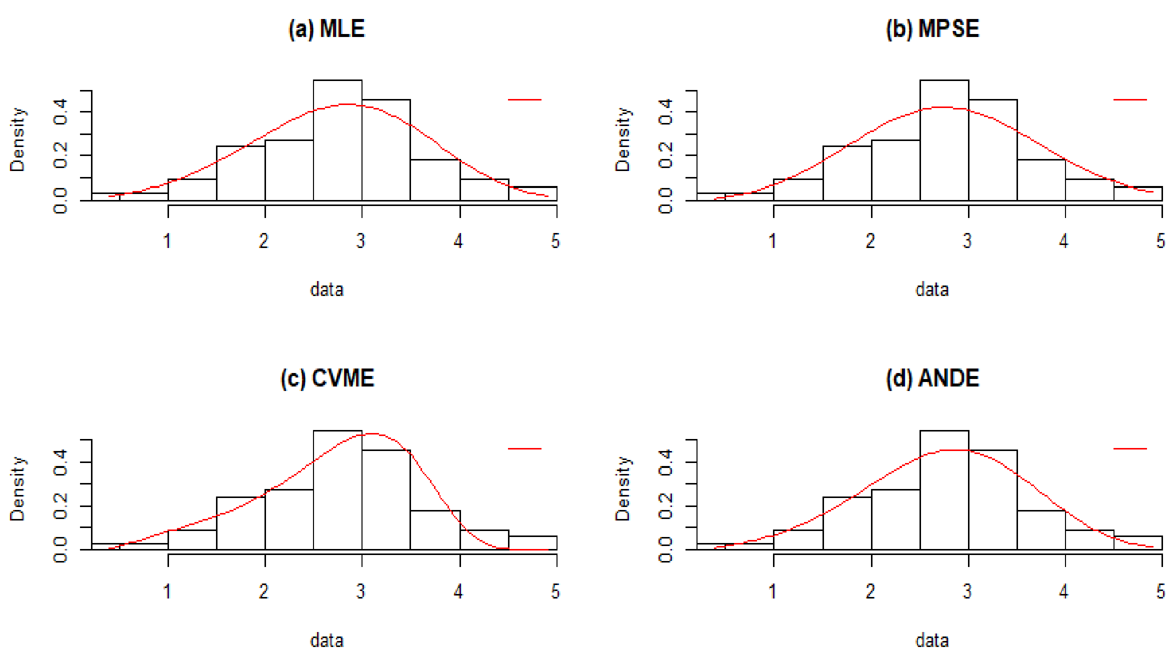

Figure 11. The EGG_{IE} fitted density function on the second data histogram with the four procedures.

7. CONCLUSIONS

This paper introduces a new four-parameter distribution called the exponentiated Gompertz generated inverted exponential (EGG_{IE}) distribution. The explicit expressions of some structural properties of the EGG_{IE} distribution are derived. The parameters of the EGG_{IE} distribution using some classical estimators are estimated. Hence, an evaluation of the four procedures in estimating parameters of the EGG_{IE} distribution through a Monte Carlo simulation using the finite sample performance. The importance of the EGG_{IE} distribution is demonstrated by fitting two real datasets, showing that the EGG_{IE} distribution provides better goodness of fit than some competing distributions considered in this study. The empirical findings indicate that the maximum likelihood procedure dominates the other estimators in the simulation study while the Cramer-Von Mises procedure dominates in the two real datasets applications. Hence, we suggest using the EGG_{IE} distribution on datasets with characteristics such as skewness and increasing hazard rates.

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APPENDIX A

Table A1. Simulation study for the EGG_{IE} distribution using the four classical estimation procedures.

n	Par	MLE				MPSE				ANDE				CVME			
		AE	**ABS**	**MSE**	**RMSE**	**AE**	**ABS**	**MSE**	**RMSE**	**AE**	**ABS**	**MSE**	**RMSE**	**AE**	**ABS**	**MSE**	**RMSE**
20	$\hat{\theta}$	1.108	0.108	0.331	0.576	1.714	0.714	17.165	4.143	2.240	1.240	34.667	5.888	1.782	0.782	3.081	1.755
	$\hat{\gamma}$	1.425	0.225	0.484	0.696	4.115	2.915	132.880	11.527	4.909	3.709	204.958	14.314	3.619	2.419	22.258	4.718
	$\hat{\beta}$	1.116	0.116	0.082	0.286	1.575	0.575	7.748	2.784	1.664	0.664	8.992	2.999	2.523	1.523	4.179	2.044
	$\hat{\alpha}$	2.556	0.056	0.774	0.880	4.259	1.759	22.082	4.699	5.539	3.039	64.612	8.038	1.215	1.285	2.273	1.508
50	$\hat{\theta}$	1.095	0.095	0.174	0.417	1.310	0.310	4.138	2.034	1.614	0.614	7.475	2.734	1.687	0.687	2.166	1.472
	$\hat{\gamma}$	1.284	0.084	0.176	0.420	1.693	0.493	8.425	2.902	1.845	0.645	7.547	2.747	2.700	1.500	7.256	2.694
	$\hat{\beta}$	1.078	0.078	0.035	0.817	1.208	0.208	3.383	1.839	1.368	0.368	4.547	2.132	2.237	1.237	2.918	1.708
	$\hat{\alpha}$	2.519	0.019	0.494	0.703	3.799	1.299	10.502	3.241	5.407	2.907	49.943	7.067	1.372	1.128	1.974	1.405
150	$\hat{\theta}$	1.059	0.059	0.055	0.235	1.012	0.012	1.312	1.163	1.711	0.711	5.438	2.332	1.580	0.580	1.434	1.198
	$\hat{\gamma}$	1.218	0.018	0.054	0.232	1.238	0.038	0.468	0.684	1.466	0.266	1.491	1.221	1.938	0.738	1.378	1.174
	$\hat{\beta}$	1.043	0.043	0.013	0.118	0.956	0.043	1.173	1.083	1.531	0.531	3.812	1.952	1.900	0.900	1.891	1.389
	$\hat{\alpha}$	2.502	0.002	0.206	0.454	3.289	0.789	3.707	1.925	3.984	1.484	17.120	4.138	1.705	0.795	1.605	1.267
300	$\hat{\theta}$	1.049	0.049	0.030	0.173	0.905	0.095	0.493	0.702	1.601	0.601	3.781	1.944	1.437	0.437	0.983	0.992
	$\hat{\gamma}$	1.202	0.002	0.025	0.157	1.171	0.029	0.082	0.286	1.331	0.131	0.131	0.575	1.616	0.416	0.495	0.704
	$\hat{\beta}$	1.032	0.032	0.008	0.088	0.867	0.133	0.435	0.659	1.476	0.476	2.873	1.695	1.629	0.629	1.231	1.120
	$\hat{\alpha}$	2.512	0.012	0.099	0.315	3.031	0.531	1.593	1.262	3.561	1.061	9.608	3.100	1.994	0.506	1.531	1.237
1000	$\hat{\theta}$	1.021	0.021	0.007	0.084	0.969	0.031	0.067	0.260	1.506	0.506	2.804	1.675	1.411	0.411	0.965	0.982
	$\hat{\gamma}$	1.200	0.000	0.007	0.087	1.189	0.011	0.013	0.113	1.280	0.080	0.117	0.342	1.408	0.208	0.159	0.399
	$\hat{\beta}$	1.013	0.013	0.003	0.052	0.961	0.039	0.047	0.216	1.412	0.412	2.094	1.447	1.485	0.485	1.062	1.031
	$\hat{\alpha}$	2.509	0.009	0.038	0.194	2.579	0.079	0.125	0.354	2.968	0.468	3.759	1.939	2.253	0.247	1.403	1.185