A Note on Triple Repetition Sequence of Domination Number in Graphs

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Abstract

A set $D \subseteq V(G)$ is a dominating set of a graph G if $\forall\, x \in V(G) \backslash D, \exists\, y \in D$ such that $xy \in E(G)$. A dominating set $D \subseteq V(G)$ is called a connected dominating set of a graph G if the subgraph $\langle D \rangle$ induced by D is connected. A connected domination number of G, denoted by $\gamma_c(G)$, is the minimum cardinality of a connected dominating set D. The triple repetition sequence denoted by $\{S_n : n \in \mathbb{Z}^+\}$ is a sequence of positive integers which is repeated thrice, i.e., $\{S_n\} = \{1, 1, 1, 2, 2, 2, 3, 3, 3, \dots\}$. In this paper, we construct a combinatorial explicit formula for the triple repetition sequence of connected domination numbers of a triangular grid graph.

Keywords: connected domination number; triangular grid graph; triple repetition sequence.

Abstrak

Suatu himpunan $D \subseteq V(G)$ adalah himpunan pendominasi graf G apabila $\forall\, x \in V(G) \backslash D, \exists\, y \in D$ sehingga $xy \in E(G)$. Suatu himpunan pendominasi $D \subseteq V(G)$ dikatakan himpunan pendominasi terhubung dari graf G apabila subgraf $\langle D \rangle$ yang diinduksi oleh D terhubung. Suatu bilangan pendominasi dari G, dinotasikan dengan $\gamma_c(G)$, adalah kardinalitas minimum dari himpunan pendominasi terhubung D. Barisan pengulangan rangkap tiga yang dinotasikan dengan $\{S_n : n \in \mathbb{Z}^+\}$ adalah suatu barisan bilangan bulat positif yang setiap sukunya berulang tiga kali, yakni, $\{S_n\} = \{1, 1, 1, 2, 2, 2, 3, 3, 3, \dots\}$. Dalam paper ini dikonstruksi suatu rumus eksplisit kombinatorial untuk barisan pengulangan rangkap tiga dari bilangan pendominasi terhubung graf grid triangular.

Kata Kunci: *bilangan pendominasi terhubung; graf grid triangular; barisan pengulangan rangkap tiga.*

1. INTRODUCTION

One of the interesting topics in graph theory is domination in graphs [1] [2] [3] [4]. The concept of domination in graphs remains intriguing and became a center of interest for many mathematicians who deals with discrete mathematics [5] [6] [7]. There are a lot of parameters had been studied since domination in graphs is being discovered. Several applications have been contributed by the concept of domination such as finding the optimum solution in social network theory [8], protection and location strategies [9], and chemical bond problems [10] [11], among others [12] [13] [14].

Let $G = G(V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$. An open neighborhood of a vertex x in G is the set $N_G(x) = N(x) = \{y \in V(G) : xy \in E(G)\}$. The closed neighborhood of a vertex x in G is the set $N_G[x] = N[x] = \{x\} \cup N(x)$. If $A \subseteq V(G)$, then the open neighborhood of A is the set $N_G(A) = N(A) = \bigcup_{x \in A} N_G(x)$. The closed neighborhood of A is given

by $N_G[A] = N[A] = A \cup N(A)$. A set $D \subseteq V(G)$ is a dominating set of G if $\forall x \in V(G) \backslash D, \exists y \in D$ such that $xy \in E(G)$, i.e., $N[D] = V(G)$. The minimum cardinality of a dominating set is called a domination number of G and it is denoted by $\gamma(G)$. A dominating set $C \subseteq V(G)$ is called a connected dominating set of G if the subgraph $\langle C \rangle$ induced by C is connected. A connected domination number of G, denoted by $\gamma_c(G)$, is the minimum cardinality of a connected dominating set $C \subseteq V(G)$. A dominating set $I \subseteq V(G)$ is called an independent dominating set of G if no two dominating vertices are adjacent, that is, the distance between two vertices is greater or equal to 2. The independent domination number of G, denoted by $i(G)$, is the smallest cardinality of an independent dominating set of the graph G.

Let $G = P_n$, $n \in \mathbb{Z}^+$, be a path of order n and length $n-1$, and let $\{G\}_{n=1}^k$ be a sequence of k paths. Then, it is clear that $\{\gamma_n(G)\}_{n=1}^k = \{\gamma_1(P_1), \gamma_2(P_2), \gamma_3(P_3), \ldots, \gamma_k(P_k)\}$. This sequence represents the sequence of domination number of a sequence of k paths with consecutive orders. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $D_n(G) = min\{\sum_{v_i \in D} i\}$. In this paper, we consider that the configuration of dominating set satisfies the condition $D_n(G)$. Now, let x and y be two distinct vertices in $D \subseteq V(G)$. The distance $d_G(x, y)$ between two vertices x and y is defined as the length of the shortest path between x and y in G. We also consider that for any two vertices $x, y \in D$ in G satisfies the condition $min[d_G(x, y)]$. In particular, if we have the sequence of $k = 6$ paths, the dominating sets configures as shown in Figure 1.
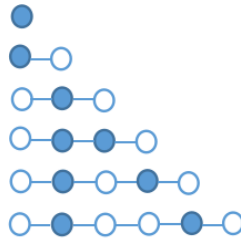

Figure 1. The sequence of the first 6 consecutive orders of paths and their dominating vertices.

A triangular grid graph with level $m \in \mathbb{Z}^+$, denoted by T_m is a subgraph of tiling on the plane with equilateral triangles defined by the finite number of triangles called cells. The order of graph T_m is $|T_m| = \frac{m(m+1)}{2}$ and $|T_m|$ is the mth triangular numbers. A triangular number is a number that counts objects organized in a form of an equilateral triangle. The details of a triangular number can be found in [15], and for more information about graph T_m, one may refer to [6] [14]. Figure 2 shows the graph T_m where $m = 7$. The triple repetition sequence denoted by $\{S_n : n \in \mathbb{Z}^+\}$ is a sequence of positive integers which is repeated thrice, i.e., $\{S_n\} = \{1, 1, 1, 2, 2, 2, 3, 3, 3, \ldots\}$. In this paper, we gave a formula for the triple repetition sequence of connected domination number of the graph T_m.

2. RESULTS AND DISCUSSION

To present our first result, we need the following remark.

Remark 2.1. [16] Let $G = P_n$ be a path of order $n \in \mathbb{Z}^+$. Then,

$$\gamma(G) = \begin{cases} \dfrac{n}{3} & if\ n \equiv 0(mod\ 3), \\ \dfrac{n+2}{3} & if\ n \equiv 1(mod\ 3), \\ \dfrac{n+1}{3} & if\ n \equiv 2(mod\ 3). \end{cases}$$
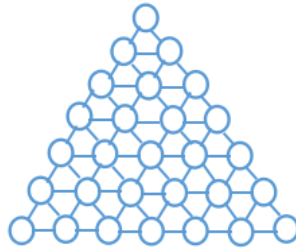

Figure 2. The graph T_7.

Example 1. Consider a path of order $n = 6$ in Figure 3. Then, we have $\gamma(G) = \frac{6}{3} = 2$.

Figure 3. Domination in the path of order $n = 6$.

Example 2. Consider a path of order $n = 7$ in Figure 4. Then, we have $\gamma(G) = \frac{7+2}{3} = 3$.

Figure 4. Domination in the path of order $n = 7$.

Example 3. Consider a path of order $n = 8$ in Figure 5. Then, we have $\gamma(G) = \frac{8+1}{3} = 3$.

Figure 5. Domination in the path of order $n = 8$.

The theorem below is a direct consequence of Remark 2.1 above. This shows that the domination number of a sequence of paths $\{G = P_n\}_{n=1}^{\infty}$ will result in a triple repetition sequence.

Theorem 2.2. Let $\{G = P_n\}_{n=1}^{\infty}$ be a sequence of paths. Then

$$\{S_n : n \in \mathbb{Z}^+\} = \{\gamma_n(G)\}_{n=1}^{\infty}.$$

Proof. Let $n \in \mathbb{Z}^+$ be the order of graph $G = P_n$. Now, if we consider $n = 1, 2, 3$, then by Remark 2.1, we obtain, $\{\gamma_n(G)\}_{n=1}^{3} = \{1, 1, 1\}$. Next, we consider adding 3 in n, that is, $n + 3$. So, we have $n + 3 = 4, 5, 6$. Again, by Remark 2.1, it implies that $\{\gamma_n(G)\}_{n=4}^{6} = \{2, 2, 2\}$. Continuing the process, it clearly follows that $\{\gamma_n(G)\}_{n=1}^{\infty} = \{1, 1, 1, 2, 2, 2, 3, 3, 3, \dots\} = \{S_n\}$. This proves the hypothesis. ∎

Example 4. Consider a sequence of paths with $n \in \{1,2,\ldots,9\}$ in Figure 6. So, we have $\{\gamma_n(G)\}_{n=1}^{9} = \{1, 1, 1, 2, 2, 2, 3, 3, 3\}$.
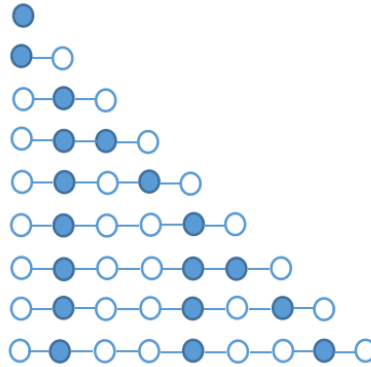

Figure 6. The sequence of paths with dominating sets.

In that case, the following corollaries are obtained as an immediate result from Remark 2.1 and Theorem 2.2. The first corollary, reveals the value of each element of the sequence $\{S_n : n \in \mathbb{Z}^+\}$. And for the second corollary, shows the formula for finding the sum of the first k elements of $\{S_n : n \in \mathbb{Z}^+\}$.

Corollary 2.3. Let $\{S_k : k \in \mathbb{Z}^+\}$ be a triple repetition sequence and $G = P_{n=k}$. Then,

$$S_k = \gamma_k(G) = \begin{cases} \dfrac{k}{3} & if\ k \equiv 0 (mod\ 3), \\ \dfrac{k+2}{3} & if\ k \equiv 1 (mod\ 3), \\ \dfrac{k+1}{3} & if\ k \equiv 2 (mod\ 3). \end{cases}$$

Proof. The proof directly follows from Theorem 2.2. ∎

Example 5. Let $k = 13$. Since $k \equiv 1 (mod\ 3)$, then, we obtain $S_{13} = \gamma_{13}(G) = \gamma_{13}(P_{13}) = \frac{13+2}{3} = 5$.

Corollary 2.4. Let $\{S_k : k \in \mathbb{Z}^+\}$ be a triple repetition sequence and let $|V(T_m)|$ be the order of the triangular grid graph, where m is a function of k, i.e., $m = \varphi(k)$. Then,

$$\sum_{n=1}^{k} S_n = \begin{cases} 3|V(T_{k/3})| & if\ k \equiv 0 (mod\ 3) \\ |V(T_{(k+2)/3})| + 2|V(T_{((k+2)/3)-1})| & if\ k \equiv 1 (mod\ 3) \\ 2|V(T_{(k+1)/3})| + |V(T_{((k+1)/3)-1})| & if\ k \equiv 2 (mod\ 3) \end{cases}$$

Proof. The proof directly follows from Theorem 2.2 and the definition of a triangular number. ∎

Example 6. Let $k = 13$. Since $k \equiv 1 (mod\ 3)$, then, we obtain

$$\sum_{n=1}^{13} S_n = |V(T_{(k+2)/3})| + 2|V(T_{((k+2)/3)-1})|$$

$$= \left|V(T_{(13+2)/3})\right| + 2\left|V(T_{((13+2)/3)-1})\right|$$

$$= |V(T_5)| + 2|V(T_4)| = 15 + 2(10) = 35.$$

In the paper of Casinillo [17], domination in path and cycle graphs was revisited, and investigated some undiscovered properties. The author has developed a new formula that determines the number of ways in putting a dominating set of vertices and provided a combinatorial proof. Let $N_\gamma(G)$ be the number of ways of putting a dominating set in a graph G. Hence, the following theorem is needed for our next result.

Theorem 2.5. [17] Let $G = P_n$ where n is a positive integer. Then,

$$N_\gamma(G) = \begin{cases} 1 & if\ n \equiv 0(mod\ 3), \\ \dfrac{n^2 + 13n + 4}{18} & if\ n \equiv 1(mod\ 3), \\ \dfrac{n + 4}{3} & if\ n \equiv 2(mod\ 3). \end{cases}$$

Example 7. Suppose that $G = P_4$, then by Remark 2.1, it follows that $\gamma(G) = \frac{4+2}{3} = 2$. Hence, by Theorem 2.5., we obtain $N_\gamma(G) = \frac{4^2+13(4)+4}{18} = 4$ (See Figure 7).
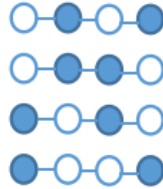

Figure 7. The four configurations of dominating set in $G = P_4$.

In that case, we can extend the result by applying it to a sequence of paths with consecutive orders. Hence, by Theorem 2.2, we obtain a new result that determines the number of ways of putting a minimum dominating set in the sequence of k consecutive order paths.

Theorem 2.6. Let $\{P_n\}_{n=1}^k$ be a sequence of k paths. If $X(n) = \frac{1}{18}(n^2 + 13n + 4)$ when $n \equiv 1(mod\ 3)$ and $Y(n) = \frac{1}{3}(n + 4)$ when $n \equiv 2(mod\ 3)$, then,

$$N_\gamma(\{P_n\}_{n=1}^k) = \begin{cases} X(1)X(4)\cdots X(k-2)Y(2)Y(5)\cdots Y(k-1) & if\ k \equiv 0(mod\ 3), \\ X(1)X(4)\cdots X(k)Y(2)Y(5)\cdots Y(k-2) & if\ k \equiv 1(mod\ 3), \\ X(1)X(4)\cdots X(k-1)Y(2)Y(5)\cdots Y(k) & if\ k \equiv 2(mod\ 3). \end{cases}$$

Proof. The proof is quick from Theorem 2.5 using the multiplicative principle. ∎

Example 8. Let $k = 4$. Since $k \equiv 1(mod\ 3)$, then, it follows that

$$N_\gamma(\{P_n\}_{n=1}^4) = X(1)X(4)Y(2) = \left[\frac{1}{18}(1^2 + 13(1) + 4)\right]\left[\frac{1}{18}(4^2 + 13(4) + 4)\right]\left[\frac{1}{3}(2 + 4)\right]$$

$$= (1)(4)(2) = 8.$$

For our next result, we use the concept of Euler's Polyhedron Formula or Euler's formula [18] for planar graph G, i.e., $|V(G)| + |F(G)| = |E(G)| + 2$. Hence, the following Lemma below is immediate.

Lemma 2.7. Let m be a positive integer. If $G = T_m$ and $H = T_{m-1}$, then

$$|F(G)| = 3|V(H)| - |V(G)| + 2 \text{ and } |E(G)| = 3|V(H)|.$$

Proof. The proof directly follows from the definition of a triangular grid graph and Euler's formula.
∎

Next, the triple repetition sequence was applied to triangular grid graphs. Firstly, we will examine the connected domination number of graph $G = T_m$. Hence, the following Lemma 2.8 will determine the connected domination number of graph $G = T_m$ where $m \in \mathbb{Z}^+$.

Lemma 2.8. Let m be a positive integer. If $G = T_m$, then

$$\gamma_c(G) = m - 1 + \sum_{m>3i}(m - 3i).$$

Proof. Suppose that $G = T_m$ where $m \in \mathbb{Z}^+$. Then, consider the sequence slanting paths of graph G and denote as $\{P_n\}_{n=1}^m$. Now, for every dominating set in the element of the sequence, we consider that it follows the condition $D_n(P_n) = min\{\sum_{v_i \in D} i\}$ and for any two vertices $u, v \in D$ must have a minimum distance, i.e., $min[d_G(u, v)]$. Hence, it follows that the dominating number of G is $m + \sum_{m>3i}(m - 3i)$ which follows the triple repetition sequence. So, this dominating set $D \subseteq V(G)$ is a connected dominating set since the subgraph $\langle D \rangle$ induced by D is connected. Since we consider the optimal solution for connected domination number, then it is concluded that $\gamma_c(G) = m - 1 + \sum_{m>3i}(m - 3i)$. This completes the proof.
∎

Example 9. Consider a graph $G = T_6$ in Figure 8. The Figure shows the configuration of dominating vertices in graph G following a triple repetition sequence in every slanting path.
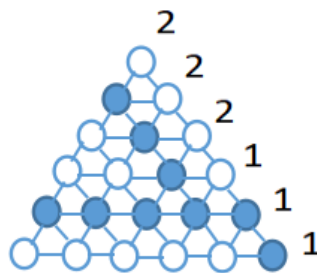

Figure 8. Graph $G = T_6$ with the connected dominating set.

Notice that in Figure 8, the cardinality of dominating set is not minimum. The dominating vertex in the last level of the graph is not needed. Hence, using Lemma 2.8, the following theorem below was constructed. The Theorem determines the connected domination number of graph $G = T_m$ where $m \in \mathbb{Z}^+ \backslash \{1, 2\}$ concerning triple repetition sequence S_n and triangular number t_n where $n \in \mathbb{Z}^+$ and n is a function of m, i.e., $n = f(m)$.

Theorem 2.9. Let n be a positive integer. If $G = T_{m \geq 3}$, then

$$\gamma_c(G) = \begin{cases} 3t_n - 1 & if \ m \equiv 0 (mod \ 3), \\ 2t_n + t_{n+1} - 1 & if \ m \equiv 1 (mod \ 3), \\ 2t_n + t_{n-1} - 1 & if \ m \equiv 2 (mod \ 3). \end{cases}$$

Proof. Let $m \geq 3$ and n be any positive integer. To prove Theorem 2.9, we need to consider the following cases below:

Case 1. First, we let $m \equiv 0 (mod \ 3)$ and $G = T_m$. Obviously, if $m = 3$, then $\gamma_c(G) = (m - 1)$. Now, consider $m > 3$. By Lemma 2.8, it follows that $\gamma_c(G) = (m - 1) + \sum_{i=1}^{\frac{m-3}{3}} (m - 3i)$.

By expanding the summation, we obtain

$$\gamma_c(G) = (m - 1) + [m - 3(1)] + [m - 3(2)] + \ldots + \left[m - 3 \left(\frac{m-3}{3} \right) \right], \text{ and}$$

$$\gamma_c(G) = (m - 1) + m \left(\frac{m-3}{3} \right) - 3 \left[1 + 2 + \ldots + \frac{m-3}{3} \right].$$

In view of a triangular number, this implies that the equation becomes

$$\gamma_c(G) = (m - 1) + \frac{1}{6}(m^2 - 3m) \text{ and } \gamma_c(G) = \frac{1}{6}(m^2 + 3m - 6).$$

Since $n = \frac{m}{3}$ and $m \equiv 0 (mod \ 3)$, then we have $m = 3n$. So, it directly follows as

$$\gamma_c(G) = \frac{1}{6}((3n)^2 + 3(3n) - 6) \text{ and } \gamma_c(G) = \frac{1}{2}(3n^2 + 3n - 2).$$

Applying the definition of triangular number, we end up with $\gamma_c(G) = 3t_n - 1$.

Case 2. Secondly, we let $m \equiv 1 (mod \ 3)$ and $G = T_m$. In view of Lemma 2.8, it follows that

$$\gamma_c(G) = (m - 1) + \sum_{i=1}^{\frac{m-2}{3}} (m - 3i).$$

Expanding the summation, we get

$$\gamma_c(G) = (m - 1) + [m - 3(1)] + [m - 3(2)] + \cdots + \left[m - 3 \left(\frac{m-2}{3} \right) \right], \text{ and}$$

$$\gamma_c(G) = (m - 1) + m \left(\frac{m-2}{3} \right) - 3 \left[1 + 2 + \cdots + \frac{m-2}{3} \right].$$

By the concept of triangular number, the equation becomes $\gamma_c(G) = \frac{1}{6}(m^2 + 3m - 4)$.

Note that $n = \left\lfloor \frac{m}{3} \right\rfloor$ and $m \equiv 1 (mod \ 3)$. Then we obtain $m = 3n + 1$. Thus,

$$\gamma_c(G) = \frac{1}{6}((3n + 1)^2 + 3(3n + 1) - 4) \text{ and } \gamma_c(G) = \frac{1}{6}(9n^2 + 15n).$$

By definition triangular number, it becomes $\gamma_c(G) = 2t_n + t_{n+1} - 1$.

Case 3. Thirdly, we let $m \equiv 2 (mod \ 3)$ and $G = T_m$. So, by Lemma 2.8, we get $\gamma_c(G) = (m - 1) + \sum_{i=1}^{\frac{m-1}{3}} (m - 3i)$. Then, by expanding the summation, we have

$$\gamma_c(G) = (m - 1) + [m - 3(1)] + [m - 3(2)] + \ldots + \left[m - 3 \left(\frac{m-1}{3} \right) \right], \text{ and}$$

$$\gamma_c(G) = (m - 1) + m \left(\frac{m-1}{3} \right) - 3 \left[1 + 2 + \ldots + \frac{m-1}{3} \right].$$

Applying the definition of triangular number, we obtain $\gamma_c(G) = \frac{1}{6}(m^2 + 3m - 4)$.

Next, it is worth noting that $n = \left\lceil \frac{m}{3} \right\rceil$ and $m \equiv 2 (mod \ 3)$. So, we obtain $m = 3n - 1$. Hence,

$$\gamma_c(G) = \frac{1}{6}((3n-1)^2 + 3(3n-1) - 4) \text{ and } \gamma_c(G) = \frac{1}{6}(9n^2 + 3n - 6).$$

So, by definition triangular number, we end up with $\gamma_c(G) = 2t_n + t_{n-1} - 1$.
Combining the three cases completes the proof. ∎

Example 10: Let $G = T_7$. Then, it follows that $n = \left\lceil \frac{7}{3} \right\rceil = 2$ and we obtain that $\gamma_c(G) = 2t_2 + t_3 - 1$
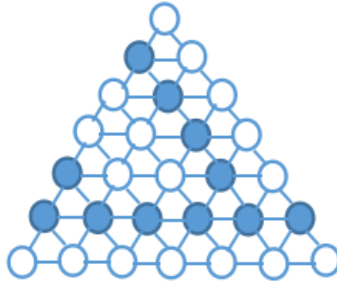$= 2(3) + 6 - 1 = 11$ as shown in Figure 9.

Figure 9. Graph T_7 with the connected dominating set.

The next Corollary is a direct consequence of Theorem 2.9 above. This shows that for some integer m, $\gamma_c(G)$ where $G = T_m$, is a function of a perfect number. In fact, there are studies in literature that deal with graphs involving the concept of perfect numbers [19] [20].

Corollary 2.10. Let n and p be prime numbers, and let $G = T_m$. If $n = \frac{m}{3}$ and $m = 3(2^p) - 3$, then $\gamma_c(G) = 3P(p) - 1$.

Proof. Let $G = T_m$. Suppose that n and p are prime numbers. Since $n = \frac{m}{3}$ and $m = 3(2^p) - 3$, then it follows that $n = 2^p - 1$. By Theorem 2.9, we have $\gamma_c(G) = 3t_n - 1$ when $m \equiv 0 \pmod 3$. Rewriting the equation, we obtain $\gamma_c(G) = 3\left[\frac{n(n+1)}{2}\right] - 1$. Substituting $n = 2^p - 1$, it follows that $\gamma_c(G) = 3\left[\frac{(2^p-1)2^p}{2}\right] - 1$. Hence, by definition of the perfect number, we end up with $\gamma_c(G) = 3P(p) - 1$. ∎

Example 11. Let $G = T_9$. It follows that $m = 3(2^2) - 3$ and $p = 2$, so, by Corollary 2.10 and definition of the perfect number, we obtain $\gamma_c(G) = 3[2^{2-1}(2^2 - 1)] - 1 = 17$.

3. CONCLUSIONS

This paper introduced a new type of sequence of numbers derived from a domination number in the sequence of path graphs $\{G = P_n\}_{n=1}^{\infty}$, i.e., the triple repetition sequence $\{S_n : n \in \mathbb{Z}^+\} = \{\gamma_n(G)\}_{n=1}^{\infty}$. It is concluded that the sum of triple repetition sequence less one is the connected domination number of triangular grid graphs T_m where $m \in \mathbb{Z}^+$. Plus, the combinatorial formula of connected domination number of triangular grid graphs T_m where $m \in \mathbb{Z}^+$ can be obtained using the concept of triple repetition sequence and a triangular number. Furthermore, the said formula can be

expressed as a function of a triangular number for all $m \in \mathbb{Z}^+\backslash\{1,2\}$ and a perfect number for some values of $m \in \mathbb{Z}^+\backslash\{1,2\}$.

REFERENCES

[1] J. L. Gross and J. Yellen, Handbook of graph theory, CRC press, 2003.

[2] R. J. Wilson, History of graph theory, Handbook of Graph Theory, 29, 2003.

[3] G. Chartrand and P. Zhang, A First Course in Graph Theory, New York: Dover Publication Inc., 2012.

[4] E. J. Cockayne and S. T. Hedetniemi, "Towards a theory of domination in graph," *Networks Advanced Topics,* vol. 7, pp. 247-261, 1977.

[5] L. F. Casinillo, E. T. Lagumbay and H. R. F. Abad, "A note on connected interior domination in join and corona of two graphs," *IOSR Journal of Mathematics,* vol. 13, no. 2, pp. 66-69, 2017.

[6] E. L. Casinillo , L. F. Casinillo and J. Valenzona, "On Triangular Secure Domination Number," *InPrime: Indonesian Journal of Pure and Applied Mathematics,* vol. 2, no. 2, pp. 105-110, 2020.

[7] M. T. K. Varkey and M. S. Thomas, "Some tree and tree related cubic graceful graphs," *International Journal of Advanced Scientific and Technical Research,* vol. 7, no. 4, pp. 239-244, 2017.

[8] A. Gagarin, A. Poghosyan and V. Zverovich, "Randomized algorithms and upper bounds for multiple domination in graphs and networks," *Discrete Applied Mathematics,* vol. 161, no. 4-5, pp. 604-611, 2013.

[9] E. C. Castillano, R. L. Ugbinada and S. R. Canoy Jr., "Secure domination in the joins of graphs," *Applied Mathematical Sciences,* vol. 8, no. 105, pp. 5203-5211, 2014.

[10] J. Quadras, A. S. M. Mahizl , I. Rajasingh and R. S. Rajan, "Domination in certain chemical graphs," *Journal of Mathematical Chemistry,* vol. 53, no. 1, pp. 207-219, 2015.

[11] E. A. Gnanasoundari, M. Raji and G. Jayalalitha, "Equitable Domination in Chemical Structural Graph," *European Journal of Molecular & Clinical Medicine,* vol. 7, no. 2, pp. 5246-5249, 2020.

[12] R. A. Borzooei and H. Rashmanlou, "Domination in vague graphs and its applications," *Journal of Intelligent & Fuzzy Systems,* vol. 29, no. 5, pp. 1933-1940, 2015.

[13] H. Rashmanlou, A. Lakdashti and S. S. Sabet, "A Study on Domination Sets in Vague Graphs with Novel Applications," *Annals of Pure and Applied Mathematics,* vol. 16, no. 1, pp. 223-234, 2018.

[14] L. F. Casinillo, "Odd and even repetition sequences of independent domination number," *Notes on Number Theory and Discrete Mathematics,* vol. 26, no. 1, pp. 8-20, 2020.

[15] L. F. Casinillo, "Some new notes on Mersenne primes and perfect numbers," *Indonesian Journal of Mathematics Education,* vol. 3, no. 1, pp. 15-22, 2020.

[16] L. F. Casinillo, "A note on Fibonacci and Lucas number of domination in path," *Electronic Journal of Graph Theory and Applications,* vol. 6, no. 2, pp. 317-325, 2018.

[17] L. F. Casinillo, "New counting formula for dominating sets in path and cycle graphs," *Journal of Fundamental Mathematics and Applications (JFMA),* vol. 3, no. 2, pp. 170-177, 2020.

[18] E. L. Casinillo and L. F. Casinillo, "A Note on Full and Complete Binary Planar Graphs," *Indonesian Journal of Mathematics Education,* vol. 3, no. 2, pp. 70-75, 2020.

[19] J. von Postel, T. Schweser and M. Stiebitz, "Point partition numbers: perfect graphs," *Graphs and Combinatorics,* vol. 38, no. 2, pp. 1-11, 2011.

[20] S. Ali, M. K. Mahmood and K. P. Shum, "Novel classes of integers and their applications in graph labeling," *Hacettepe Journal of Mathematics and Statistics,* vol. 50, no. 4, pp. 1-17, 2021.