Stability Analysis of Leslie-Gower Model with Herd Behavior on Prey

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Abstract

We studied the Leslie-Gower model of predator-prey with herd behavior. The square root functional response models predator and prey interactions that show herd behavior. This study aims to determine the formulation of the predator-prey model with herd behavior on prey, knowing the fixed points and its stability and simulating the model numerically. We found three fixed points that may exist: the extinction point of both species, the extinction of predator point, and the point of coexistence of the two species. The extinction of predator points is always unstable, while the point of coexistence of the two species can be stable under certain conditions. Due to the presence of square roots, the behavior of the solutions near the extinction point of the two species is not readily apparent. Numeric simulation shows that changing the initial condition and parameters can change the system's stability.

Keywords: predator-prey; functional response; herd behavior; square root functional response, Leslie-Gower model.

Abstrak

Artikel membahas model predator prey Leslie-Gower *dengan perilaku bergerombol pada* prey. *Interaksi* predator *dan prey yang menunjukkan perilaku bergerombol dimodelkan dengan fungsi respon akar kuadrat. Penelitian ini bertujuan untuk mengetahui formulasi model predator-prey dengan perilaku bergerombol pada prey, mengetahui titik ekuilibrium dan kestabilannya serta menyimulasikan model tersebut secara numerik. Hasil menunjukkan terdapat tiga titik tetap yang mungkin eksis, yaitu titik kepunahan kedua spesies, titik kepunahan predator dan titik koeksistensi kedua spesies. Titik kepunahan predator selalu tidak stabil, sedangkan titik koeksistensi kedua spesies bisa stabil dengan syarat tertentu. Karena adanya akar kuadrat, perilaku solusi di dekat titik kepunahan kedua spesies tidak mudah terlihat. Simulasi numerik menunjukkan bahwa perubahan nilai awal dan parameter dapat mengubah kestabilan sistem.*

Kata Kunci: predator prey; *fungsi respons*; *perilaku bergerombol*; *fungsi respon akar kuadrat*; *model* Leslie-Gower.

1. INTRODUCTION

Predator-prey model is an essential tool for understanding the interactions between species in a natural environment. The first model describing the interactions between two species, prey, and predator, was introduced by Lotka and Volterra, which became known as the Lotka-Volterra predator-prey model [1] [2] [3]. The model is a system of ordinary differential equations. The predation function of predators on prey in this model is linear or called Holling type I. Other common functional responses for predator-prey models are types II, III, and IV [4] [5] [6] [7]. Leslie and Gower modified the model by adding carrying capacity for the predator population in proportion to the size or number of the prey population [8]. The predator-prey model often studied is the Leslie-Gower model with a linear response function as in the Collings [9]. This model is:

$$\frac{dX}{dt} = r_1 X \left(1 - \frac{X}{K}\right) - aXY,$$

$$\frac{dY}{dt} = r_2 Y \left(1 - \frac{Y}{bX}\right),$$

(1)

where $r_1, r_2, a, b, K > 0$, r_1 and r_2 represent the intrinsic growth of predator and prey, respectively, a describes the rate of predation, and b describes the growth rate of the predator as a result of predation. Aziz-Alaoui and Okiye [10] modified Leslie Gower's predator-prey model by adding the amount of environmental protection to maintain the survival of predators. The model is as follows:

$$\frac{dX}{dt} = r_1 X \left(1 - \frac{X}{K}\right) - \frac{aX}{m_1 + X} Y,$$

$$\frac{dY}{dt} = r_2 Y \left(1 - \frac{Y}{m_2 + X}\right),$$

(2)

where $m_1, m_2, r_1, r_2, a, K > 0$, m_1 and m_2 describe the environmental protection of predator and prey populations, respectively.

Animals often have herd behavior as a form of self-defense from predators. As a result, predators can only attack prey outside the pack. As in animals in a savanna, predators can only attack the outside of the swarming prey group. Or the fish that cluster in the sea to defend themselves from shark attacks. That social behavior in the prey population often appears because they live in a herd. The individuals living on the outskirts can protect those residing at the flock's center. Such a model is considered by Ajraldi et al. [11], assuming that the interactions occur along the boundary of the population. They use the square root of the prey population in interaction terms to describe herd behavior.

Braza [12] also described herd behavior in prey with the square root response function. The proposed model is to modify the Lotka-Volterra model using the Holling type II and square root functional responses as follows

$$\frac{dX}{dt} = r_1 X \left(1 - \frac{X}{K}\right) - \frac{a\sqrt{X}}{1 + \sqrt{X}} Y,$$

$$\frac{dY}{dt} = -sY + \frac{b\sqrt{X}Y}{1 + a\sqrt{X}},$$

(3)

where s is the death rate of the predator. He has shown that the square root term makes the solution behavior near the origin more subtle than standard models. In this article, we studied the Leslie-Gower model of predator-prey with herd behavior. We determine the formulation of the predator-prey model with herd behavior on prey, knowing the fixed points and their stability. In addition, at the end of the article, we present a numerical simulation of the model.

2. METHODS

We modified the two-species Leslie-Gower model of the system (1) by adding square root functional response as in system (3) to describe herd behavior in prey. So that the model becomes

$$\frac{dX}{dt} = r_1 X \left(1 - \frac{X}{K}\right) - a\sqrt{X}Y, \tag{4a}$$

$$\frac{dY}{dt} = r_2 Y \left(1 - \frac{Y}{b\sqrt{X}}\right), \tag{4b}$$

where each parameter is
positive. Table 1 shows the variables and parameters used in our model.

Table 1: Model variables and parameters

Symbol	Description
X	Number of prey
Y	Number of predators
t	Time
r_1	Intrinsic growth of prey
r_2	Intrinsic growth of predator
a	Rate of predation
b	The growth rate of the predator as a result of predation
K	Carrying capacity

First, we calculate the system's fixed points and then perform a local stability analysis of those points by linearization [13]. The next step is to do some numerical simulations using the Maple program to see the system's behavior by changing a parameter and initial values. Finally, we explain the results of the numerical simulations and conclude the system.

3. RESULTS AND DISCUSSIONS

3.1 Fixed Points of the System and Its Stability

The system's fixed point (4) is obtained by solving the equation $\frac{dX}{dt} = 0$ dan $\frac{dY}{dt} = 0$. There are three fixed-point solutions:

1. $E_1 = (0,0)$, which illustrates that both species are extinct,
2. $E_2 = (K, 0)$, which illustrates that only the prey species have survived,
3. $E_3 = \left(\frac{K(r_1 - ab)}{r_1}, b\sqrt{\frac{K(r1 - ab)}{r1}}\right)$ where $K - \frac{Kab}{r_1} > 0$, a fixed point where prey and predator coexist.

The stability of this fixed point determine using the eigenvalues of the following Jacobian matrix,

$$\begin{bmatrix} r_1 - \frac{2r_1 X}{K} - \frac{aY}{2\sqrt{X}} & -a\sqrt{X} \\ \frac{r_2 Y^2}{2bX\sqrt{X}} & r_2 - \frac{2r_2 Y}{b\sqrt{X}} \end{bmatrix}. \tag{5}$$

This Jacobian matrix is evaluated for each fixed point. For point $E_1 = (0,0)$ the characteristic equation cannot be determined because the values of $\frac{r_2 Y^2}{2bX\sqrt{X}}$ for $X = 0$ and $Y = 0$ are undefined. The system behavior around point E_1 will be explained in the numerical simulation. Furthermore, the system behavior in E_2 and E_3 are as follows.

1. **A fixed point where only prey species exist**

   The Jacobian matrix at point E_2 is

$$J(E_2) = \begin{bmatrix} -r_1 & -a\sqrt{K} \\ 0 & r_2 \end{bmatrix}.$$

   The eigenvalues of $J(E_2)$ are $\lambda_1 = -r_1$ and $\lambda_2 = r_2$, which are the saddle points. Since each parameter is positive, point E_2 is always unstable.

2. **A fixed point where predator and prey co-exist**

   The Jacobian matrix at point E_3 is

$$J(E_3) = \begin{bmatrix} -r_1 + \dfrac{3}{2}ab & -a\sqrt{\dfrac{K(r_1 - ab)}{r_1}} \\ \dfrac{r_2 b}{2\sqrt{\dfrac{Kr_1 - ab}{r_1}}} & -r_2 \end{bmatrix}$$

   The characteristic equation of the matrix $J(E_3)$ is $\lambda^2 + \left(r_1 + r_2 - \frac{3}{2}ab\right)\lambda + r_1 r_2 - abr_2$. So that the eigenvalue of $J(E_3)$ is

$$\lambda_{1,2} = \frac{-\left(r_1 + r_2 - \dfrac{3}{2}ab\right) \pm \sqrt{D}}{2},$$

   with $D = \left(r_1 + r_2 - \frac{3}{2}ab\right)^2 - 4(r_1 r_2 - abr_2)$. Point E_3 is stable if both eigenvalues are negative, that is, when $D \geq 0$, $\lambda_1 + \lambda_2 < 0$, and $\lambda_1 \cdot \lambda_2 > 0$.

3.2 Numerical Simulations

This section presents some numerical simulation results that illustrate the system's behavior (4) by varying the system's parameters. In Figure 1, we use the parameters $r_1 = 0.8; r_2 = 0.3; K = 5; a = 0.5$ and $b = 1$. Three fixed points exist i.e. the point of extinction for the two species E_1, which is unstable; the point of extinction for predator E_2, which is also unstable; and the point E_3 which shows the two species coexist, which is asymptotically stable. Points E_1 and E_2 can be saddle points because each trajectory approaches the two points and then moves away at a certain initial value, while point E_3 is a spiral sink.

Furthermore, using the parameters $r_1 = 2; r_2 = 0.5; K = 5; a = 0.7;$ and $b = 1$. The system behavior also shows results similar to Figure 2. There are three points that exist: the unstable extinction point of the two species, the unstable predator extinction point, and the coexistence of the two species point that is asymptotically stable.

In the third simulation, the parameters $r_1 = 0.5; r_2 = 0.8; K = 0.5; a = 0.8$, and $b = 0.7$. In Figure 3, it can be seen that only 2 equilibrium points exist because the value of $K - \frac{Kab}{r_1} = -0.06 < 0$, so E_3

does not exist. The extinction point of the two species E_1 is stable, and the predator extinction point E_2 is unstable.
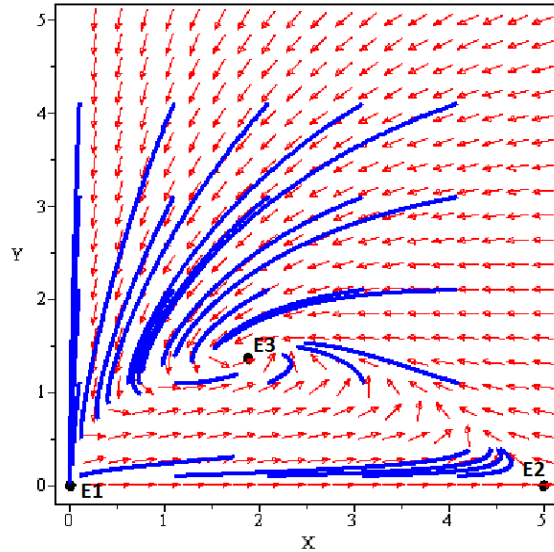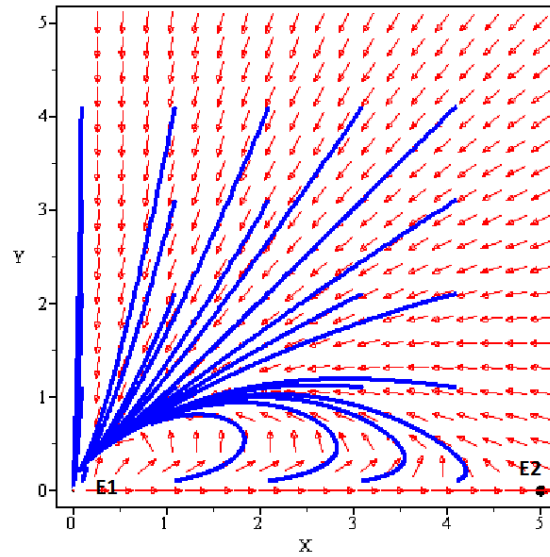

**Figure 1.** There are three equilibrium points that exist, E_1, E_2, and E_3.
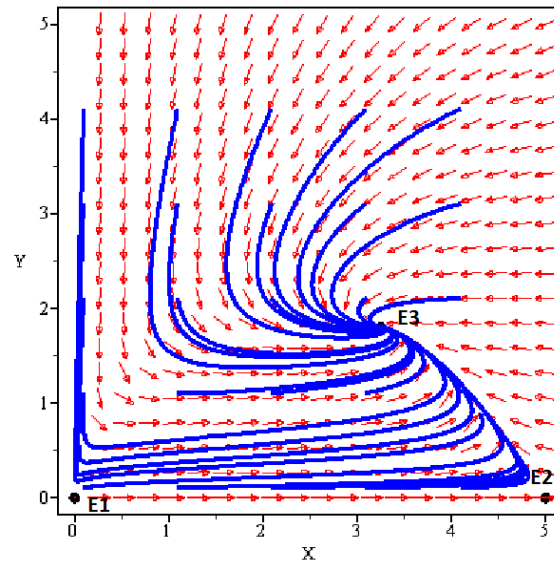
Figure 2. Parameter change shifts point E_3 but do not affect its stability.

In Figure 4, the parameters used are $r_1 = 0{,}5; r_2 = 0{,}01; K = 5; a = 0{,}8$, and $b = 0{,}05$. There are three fixed points in existence. The stability of points E_1 and E_3 depends on the starting point or initial value used. While pointing out E_2 is unstable.

Figure 3. The parameters do not meet $K - \frac{Kab}{r_1} < 0$, so E_3 does not exist.
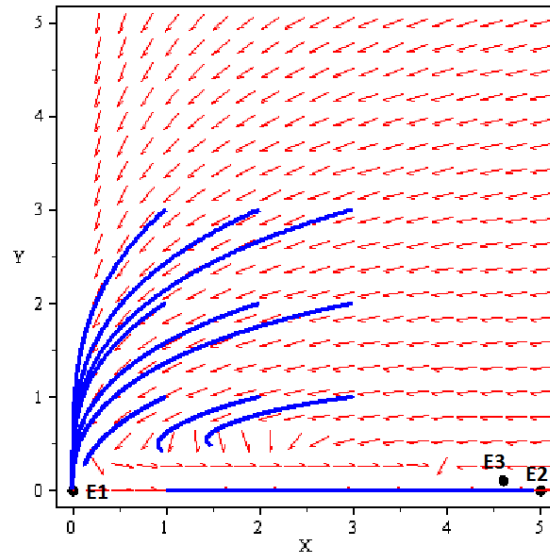
Figure 4. The stability of points E_1 and E_3 depend on the initial value, while point E_2 is unstable.

4.   CONCLUSIONS

This paper studied the Leslie-Gower model with a square root functional response that describes herd behavior on prey. We have shown that three equilibrium points exist, E_1, E_2, and E_3. We are also shown that E_2 is always unstable and E_3 will be stable if $\left(r_1 + r_2 - \frac{3}{2}ab\right)^2 - 4(r_1 r_2 - abr_2) \geq 0$, $\lambda_1 + \lambda_2 < 0$ and $\lambda_1 \cdot \lambda_2 > 0$. The stability of E_1 cannot be determined by linearization. Based on the numeric simulation, the stability of E_1 depends on the initial condition. We have observed that the square root functional response can make the system near the origin more subtle. If the prey

population is smaller than the predator, both species will reach extinction, and if the predator has a smaller population, the system will be away from the origin. This makes an ecological balance. Herd behavior on prey is like self-defense to avoid predation. The predator can only attack the outside of the herd.

In this study, we have analyzed the stability of the system locally. For further research, we suggest analyzing the occurrence of bifurcation in the system and its stability globally. This system modification is also implemented by changing the system to add other variables so that it becomes 3 species system.

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