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Existence and Uniqueness of Fixed Point for Cyclic Mappings in *Quasi*-αb-Metric Spaces

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Abstract

The fixed point theory remains the most important and preferred topic studied in mathematical analysis. This study discusses sufficient conditions to prove a unique fixed point in quasi-αb-metric spaces with cyclic mapping. The analysis starts by showing fulfillment of the cyclic Banach contraction and proving the Cauchy sequence as a condition for proving a unique fixed point in quasi-αb-metric spaces with cyclic mapping. Furthermore, it's shown that the cyclic mappings, T have a unique fixed point in quasi-αb-metric spaces. Finally, an example is given to strengthen the proof of the theorems that have been done.

Keywords: fixed point theory; Quasi αb-Metric spaces; Cyclic Banach Contraction; Cauchy sequence.

Abstrak

Teori titik tetap termasuk salah satu topik penting dan menarik untuk diteliti pada bidang analisis. Pada penelitian ini, dibahas tentang syarat cukup dalam membuktikan bahwa terdapat titik tetap tunggal dalam ruang quasi-α*b-metrik pada pemetaan siklik. Analisis diawali dengan menunjukkan pemenuhan kondisi kontraksi* Banach *siklik dan pembuktian barisan* Cauchy *sebagai syarat pembuktian bahwa terdapat titik tetap tunggal pada pemetaan siklik dalam ruang* quasi-αb-metrik. *Selanjutnya ditunjukkan bahwa pemetaan siklik* T *memiliki titik tetap tunggal dalam ruang quasi* α*b-metrik. Terakhir, diberikan contoh untuk memperkuat pembuktian teorema yang telah dilakukan.*

Kata Kunci: *teori titik tetap*; *ruang* Quasi αb-Metrik; Kontraksi Banach Siklik; *barisan* Cauchy.

1. INTRODUCTION

The analysis is one of the scopes of study in Mathematics. In analysis, research topics can be studied, including the fixed point theory, the normed spaces, the topological spaces, the Hilbert spaces, and others. Of the several research topics mentioned, the fixed point theory is one most important and preferred topics to be studied [1].

In 1992, Banach proved a unique fixed point in complete metric spaces for contraction mappings, known as Banach's fixed point theory [2],[3]. The theorems have ensured fixed point existence and uniqueness with functions defined to complete space and contractive function [4]. The theory of fixed point has a significant role in solving the problems in mathematics, i.e., linear equations, differential equations (ordinary and partial), and integral equations [5]. Fixed point theory has also helped solve problems in other scopes such as biology, physics, chemistry, economics, programming, and electronic engineering [6].

In recent years, there have been many researchers discussing Banach's fixed point theory, e.g., the metric spaces [7]-[10], b-metric spaces [11]-[14], and αb-metric spaces [15]-[18]. From the studies

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above, it's proven that various mapping types have a unique fixed point. Studies related to metric space are still open to be carried out now. This article will study the sufficient conditions to prove that quasi-αb-metric spaces have some unique fixed point for cyclic mappings.

2. PRELIMINARY

Definition 2.1. [19],[20] *Let* $\Re$ *be define a non-empty set for* $k \in [1, \infty)$. *Let* $d_c : \Re \times \Re \to [0, \infty)$ *defined as mapping and for all* $r, s, t \in \Re$, *has to satisfy the conditions below:*

(1) $\mathrm{d_c(r,s) \geq 0}$
(2) $\mathrm{d_c(r,s) = d_c(s,r) = 0}$, if only if $\mathrm{r = s}$
(3) $\mathrm{d_c(r,s) = d_c(s,r)}$
(4) $\mathrm{d_c(r,s) = k.[d_c(r,t) + d_c(t,s)]}$

Where d_c *is defined as b-metric in* $\Re$. *If (1)-(4) hold, then* $(\Re, d_c)$ *is defined as b-metric space in* $\Re$. *If (1)-(2) and (4) holds, then* $(\Re, d_c)$ *is defined as quasi b-metric space in* $\Re$ *and if (2)-(4) hold, then* $(\Re, d_c)$ *is defined as dislocated b-metric space in* $\Re$.

Definition 2.2. [17] *Let* $\Re$ *be define a non-empty set for* $\alpha \in [0,1)$ *and* $b \in [1, \infty)$. *Let* $d_c : \Re \times \Re \to [0, \infty)$ *defined as mapping and for all* $r, s, t \in \Re$, *has to satisfy the conditions below:*

(1) $d_c(r,s) \geq 0$
(2) $d_c(r,s) = d_c(s,r) = 0$, *if only if* $r = s$
(3) $d_c(r,s) = d_c(s,r)$
(4) $d_c(r,s) = \alpha . d_c(s,r) + \frac{1}{2} b[d_c(r,t) + d_c(t,s)]$

Where d_c *is defined as* α*b-metric in* $\Re$. *If (1)-(4) holds, then* $(\Re, d_c)$ *is defined as* α*b-metric space in* $\Re$. *If (1)-(2) and (4) hold, then* $(\Re, d_c)$ *is defined as quasi-*α*b-metric space in* $\Re$ *and if (2)-(4) hold, then* $(\Re, d_c)$ *is defined as dislocated-*α*b-metric space in* $\Re$.

Definition 2.3. [17] *Let* $(\Re, d_c)$ *defined as quasi-*α*b-metric space. Let* $\{p_n\}$ *defined as sequence in* $(\Re, d_c)$ *is converges in* $p \in \Re$ *if the condition* $\lim_{n \to \infty} d(p_n, p) = \lim_{n \to \infty} d(p, p_n) = 0$ *hold. And we write* $\lim_{n \to \infty} p_n = p$.

Definition 2.4. [17] *Let* $\{p_n\}$ *defined as sequence in quasi-*α*b-metric space* $(\Re, d_c)$. $\{p_n\}$ *defined the Cauchy sequence if the condition* $\lim_{n,m \to \infty} d(p_n, p_m) = \lim_{n,m \to \infty} d(p_m, p_n) = 0$ *holds.*

Definition 2.5. [17] *Let* $(\Re, d_c)$ *is defined as quasi-*α*b-metric space.* $(\Re, d_c)$ *is said as complete if for all Cauchy sequence* $\{p_n\} \subset \Re$ *converges in* $\Re$.

Lemma 2.6. [15] *Let* $(\Re, d_c)$ *is defined as quasi-*α*b-metric space for* $\alpha \in [0,1)$ *and* $b \in [1, \infty)$. *Let* T *be defined as self-mapping on* $\Re$ *if the condition of Banach-contraction below*

$$d_c(Tr, Ts) \leq \lambda . d_c(r,s),$$

hold for $r, s \in \Re$, $\lambda \in (0,1)$. *Then* T *is continuous mapping on* $\Re$.

Theorem 2.7. [15] *Let* $(\Re, d_c)$ *defined a quasi-*α*b-metric space for* $\alpha \in [0,1)$ *and* $b \in [1, \infty)$ *and let* $\{p_n\}$ *defined as a sequence in* $\Re$ *if the conditions below:*

(1) $d_c(p_n, p_{n+1}) \leq \beta . d_c(p_{n-1}, p_n)$ *for* $\beta \in (0,1)$;
(2) $\mathrm{d_c(p_{n+1}, p_n) \leq \gamma . d_c(p_n, p_{n-1})}$ for $\gamma \in (0,1)$;
(3) $\mathrm{b\beta + \alpha^2 < 1}$ and $\mathrm{b\gamma + \alpha^2 < 1}$

Hold. Then $\{p_n\}$ *is a Cauchy sequence in* $\Re$.

Theorem 2.8. [15] *Let* $(\Re, d_c)$ *is defined as a complete quasi* α*b-metric space for* $\alpha \in [0,1)$ *and* $b \in [1,\infty)$. *Let* T *be defined as self-mapping on* $\Re$ *that the condition of Banach-contraction below*

$$d_c(Tr,Ts) \leq \lambda.\, d_c(r,s),$$

hold for $r,s \in \Re$, $\lambda \in (0,1)$ *and* $b\lambda + \alpha^2 < 1$. *Then* T *has a unique fixed point in* $\Re$.

Theorem 2.9. [17] *Let* $(\Re, d_c)$ *defined as quasi* α*b-metric space for* $\alpha \in [0,1)$ *and* $b \in [1,\infty)$. *Then for* $r,s,t \in \Re$ *holds*

(1) $d_c(r,s) \leq \frac{b}{2(1-\alpha^2)}\{d_c(r,t) + d_c(t,s) + \alpha[d_c(s,t) + d_c(t,s)]\}$

(2) $\mathrm{d_c(r,s) + d_c(s,r)} \leq \frac{\mathrm{b}}{2(1-\alpha)}[\mathrm{d_c(r,t) + d_c(t,s) + d_c(s,t) + d_c(t,r)}]$

Definition 2.10. [21][22] *Let* A *and* B *be define a non-empty set of metric space* $(\Re, d_c)$ *and* $T : A \cup B \to A \cup B$. *Mapping* T *is defined as cyclic mapping if the condition* $T(A) \subseteq B$ *and* $T(B) \subseteq A$ *holds.*

Theorem 2.11. [23],[24] *Let* $\sum a_n$ *be a series of positive numbers and*

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho,$$

(1) *If* $\rho < 1$, *then the series converges.*
(2) *If* $\rho > 1$, *then the series diverges.*
(3) *If* $\rho = 1$, *then inconclusive.*

3. RESULTS AND DISCUSSIONS

This section will discuss a lemma and theorem concerning the uniqueness of fixed points for cyclic mapping in *quasi* αb-metric spaces.

Lemma 3.1. *Let* $(\Re, d_c)$ *defined as quasi-*α*b-metric space for* $\alpha \in [0,1)$, $b \in [1,\infty)$ *and* $\frac{b}{2(1-\alpha)} < 1$. *If* $\{p_n\} \subseteq \Re$ *defined as sequence in quasi-*α*b-metric space* $(\Re, d_c)$ *that satisfies*

$$d_c(p_n,p_{n+1}) + d_c(p_{n+1},p_n) \leq k.\,[d_c(p_{n-1},p_n) + d_c(p_n,p_{n-1})].$$

For $\frac{b}{2(1-\alpha)} \leq k < 1$. *Then* $\{p_n\}$ *is called a Cauchy sequence.*

Proof:

According to **Definition 2.4.**, we prove that $\{p_n\}$ is a Cauchy sequence in *quasi-*αb-metric space $(\Re, d_c)$. Taken $m,n \in \mathbb{N}$. Let $n < m$ and $r = \frac{b}{2(\alpha-1)}$. According to **Theorem 2.9.**, we obtain

$$\begin{aligned} d_c(p_m,p_n) + d_c(p_n,p_m) &\leq r.\begin{bmatrix} d_c(p_m,p_{m-1}) + d_c(p_{m-1},p_m) \\ +d_c(p_{m-1},p_n) + d_c(p_n,p_{m-1}) \end{bmatrix} \\ &= r.\,[d_c(p_m,p_{m-1}) + d_c(p_{m-1},p_m)] \\ &\quad +r.\,[d_c(p_{m-1},p_n) + d_c(p_n,p_{m-1})] \\ &\leq r.\,[d_c(p_m,p_{m-1}) + d_c(p_{m-1},p_m)] \end{aligned}$$

$$
\begin{aligned}
&+r^2.[d_c(p_{m-1},p_{m-2})+d_c(p_{m-2},p_{m-1})] \\
&+r^2.[d_c(p_n,p_{m-2})+d_c(p_{m-2},p_n)] \\
\leq\ & r.[d_c(p_m,p_{m-1})+d_c(p_{m-1},p_m)] \\
&+r^2.[d_c(p_{m-1},p_{m-2})+d_c(p_{m-2},p_{m-1})] \\
&+\cdots+r^{m-n-1}.[d_c(p_{n+2},p_{n+1})+d_c(p_{n+1},p_{n+2})] \\
&+r^{m-n-1}.[d_c(p_{n+1},p_n)+d_c(p_n,p_{n+1})].
\end{aligned}
$$

From the inequality above, an infinite series will be formed as follows:

$$
\begin{aligned}
d_c(p_m,p_n)+d_c(p_n,p_m) &= \sum_{i=1}^{m-n-1} r^i[d_c(p_{m-i+1},p_{m-i})+d_c(p_{m-i},p_{m-i+1})] \\
&\quad +r^{m-n-1}[d_c(p_{n+1},p_n)+d_c(p_n,p_{n+1})] \\
&\leq \sum_{i=1}^{m} k^i\left[k^{m-i-1}[d_c(p_1,p_2)+d_c(p_2,p_1)]\right] \\
&\quad +k^{m-n-i}\left[k^{n-i}[d_c(p_1,p_2)+d_c(p_2,p_1)]\right] \\
&= \sum_{i=1}^{m}\left[k^{m-1}[d_c(p_1,p_2)+d_c(p_2,p_1)]\right] \\
&\quad +\left[k^{m-2}[d_c(p_1,p_2)+d_c(p_2,p_1)]\right] \\
&= \sum_{i=1}^{m}(1).\left[k^{m-1}[d_c(p_1,p_2)+d_c(p_2,p_1)]\right] \\
&\quad +\left[k^{m-2}[d_c(p_1,p_2)+d_c(p_2,p_1)]\right] \\
&= (m).\left[k^{m-1}[d_c(p_1,p_2)+d_c(p_2,p_1)]\right] \\
&\quad +\left[k^{m-2}[d_c(p_1,p_2)+d_c(p_2,p_1)]\right] \\
&= (mk^{m-1}+k^{m-2})[d_c(p_1,p_2)+d_c(p_2,p_1)]
\end{aligned}
$$

So, we write

$$
\lim_{n,m\to\infty}[d_c(p_m,p_n)+d_c(p_n,p_m)] = \lim_{n,m\to\infty}(mk^{m-1}+k^{m-2}).[d_c(p_1,p_2)+d_c(p_2,p_1)]
$$

By using the ratio test according to **Theorems 2.11.**, we obtain

$$\lim_{n,m\to\infty}[d_c(p_m,p_n)+d_c(p_n,p_m)] = 0.[d_c(p_1,p_2)+d_c(p_2,p_1)]$$

$$\lim_{n,m\to\infty}[d_c(p_m,p_n)+d_c(p_n,p_m)] = 0$$

So, $\{p_n\}$ is a Cauchy sequence in *quasi*-αb-metric space in $\mathfrak{R}$.■

Next, we give a theorem about the properties of cyclic Banach contraction mappings.

Theorem 3.2. *Let* $(\mathfrak{R},d_c)$ *is defined as quasi-*α*b-metric space for* $\alpha\in[0,1)$, $b\in[1,\infty)$, $\frac{b}{2(1-\alpha)}<1$ *and* $\{\mathcal{H}_i\}_{i=1}^m$ *defined as a family set of a non-empty set of* $(\mathfrak{R},d_c)$. *If mapping* $T: \bigcup_{i=1}^m\mathcal{H}_i\to\bigcup_{i=1}^m\mathcal{H}_i$ *holds*

(1) $T(\mathcal{H}_i)\subseteq\mathcal{H}_{i+1}$, *for every* $1\le i\le m$ *where* $\mathcal{H}_{m+1}=\mathcal{H}_1$;
(2) There is k *where* $\frac{b}{2(1-\alpha)}<k<1$ *that for every* $p\in\bigcup_{i=1}^m\mathcal{H}_i$, *hold*

$$d_c(T^2p,Tp)+d_c(Tp,T^2p)\le k[d_c(Tp,p)+d_c(p,Tp)].$$

Then $\bigcap_{i=1}^m\mathcal{H}_i\neq\emptyset$.

Proof:

Taken $q\in\bigcup_{i=1}^m\mathcal{H}_i$. From inequality in condition (2) above, we obtain

$$\begin{aligned} d_c(T^{n+1}q,T^nq)+f(T^nq,T^{n+1}q) &= d_c\big(T(T^nq),T(T^{n-1}q)\big)+d_c(T(T^{n-1}q),T(T^nq)) \\ &\le k[d_c(T^nq,T^{n-1}q)+d_c(T^{n-1}q,T^nq)] \end{aligned}$$

According to **Lemma 3.1.**, $\{T^nq\}$ is called a Cauchy sequence. We have known that $(\mathfrak{R},d_c)$ complete and $(\bigcup_{i=1}^m\mathcal{H}_i,d_c)$ also complete. So that $\{T^nq\}$ is convergent to $r\in\bigcup_{i=1}^m\mathcal{H}_i$. Further, from condition (1), $\{T^nq\}\in\mathcal{H}_i$ for every i where $1\le i<m$ and $\mathcal{H}_i$ is a non-empty set for every i where $1\le i<m$ we obtain $r\in\bigcap_{i=1}^m\mathcal{H}_i$ that means $\bigcap_{i=1}^m\mathcal{H}_i\neq\emptyset$.■

After we obtained the theorem about properties of cyclic Banach contraction mappings. In the next step, we give a theorem about cyclic Banach type fixed point in *quasi*-αb-metric spaces.

Theorem 3.3. *Let* $(\mathfrak{R},d_c)$ *defined as quasi-*α*b-metric space for* $\alpha\in[0,1)$, $b\in[1,\infty)$ *and* $\frac{b}{2(1-\alpha)}<1$. *Let* A *and* B *be define a non-empty sets of quasi-*α*b-metric space* $(\mathfrak{R},d_c)$ *and cyclic mappings* $T:A\cup B\to A\cup B$ *that satisfies cyclic-Banach-contraction*

$$d_c(Tp,Tq)\le k.d_c(p,q).$$

For $p\in A$, $q\in B$, $\frac{b}{2(1-\alpha)}<1$ *and* $bk+\alpha^2<1$. *Then* T *there exists a unique fixed point in* $A\cap B$.

Proof:

Taken $p\in A$, $Tp\in B$ and $\{p_n\}$ defined as sequence in $\mathfrak{R}$ where $\mathfrak{R}_{n+1}=Tp_n$ for $n\ge 0$. Since $A\cap B\subseteq\mathfrak{R}$, then $A\cap B\subseteq\{p_n\}$. By using the cyclic Banach contraction above, we obtain

$$d_c(p_n,p_{n+1})=d_c(Tp_{n+1},Tp_n)\le k.d_c(p_{n-1},p_n),$$

and

$$d_c(p_{n+1}, p_n) = d_c(Tp_n, Tp_{n+1}) \leq k. d_c(p_n, p_{n-1}).$$

According to **Theorem 3.2.**, we obtain that $A \cap B \neq \emptyset$. Further, since $A, B \subseteq \mathfrak{R}$ where $(\mathfrak{R}, d_c)$ complete and $A \cap B$ are closed, $(A \cap B, d_c)$ are also complete. On the other side, a consequence of T satisfies cyclic-Banach-contraction then $T(A \cap B) \subseteq (A \cap B)$ and

$$d_c(Tf, Tg) \leq k. d_c(f, g).$$

For every $f, g \in A \cap B$. According to **Theorem 2.8.**, we obtain that T has a unique fixed point in $A \cap B$.■

4. CONCLUSIONS

The cyclic mappings T in a complete quasi-αb-metric space with $0 \leq \alpha < 1,\ b \geq 1$ and $\frac{b}{2(1-\alpha)} < 1$ has a unique fixed point if cyclic-Banach-contraction condition below

$$d_c(Tr, Ts) \leq k. d_c(r, s).$$

Hold. For $r \in A$, $s \in B$, $\frac{b}{2(1-\alpha)} \leq k < 1$ and $bk + \alpha^2 < 1$.

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