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# E-Cordial Labeling for Cupola Graph Cu(3, b, n)

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**Abstract**

Graph labeling is a map that assigns graph elements such as vertices, edges, vertices, and edges to a set of numbers. A graph labeling is named e-cordial if there is a binary mapping $f: E(G) \rightarrow \{0,1\}$ which induces the vertex labeling defined by $g(v) = \sum_{uv \epsilon E(G)} f(uv)(mod\ 2)$, so that it satisfies the absolute value of the difference between the number of vertices labeled 1 and the number of vertices labeled 0 is less than equal to 1, and also for the number of edges labeled 0 and labeled 1. A graph that admits the e-cordial labeling is called an e-cordial graph. This paper proved that some of the cupola graph $Cu\ (3, b, n)$ is e-cordial.
**Keywords:** E-Cordial Labeling; E-Cordial Graph; Cupola Graph Cu(a, b, n).

**Abstrak**

*Pelabelan graf merupakan pemetaan yang memetakan unsur-unsur graf seperti simpul, sisi, simpul dan sisi ke himpunan bilangan. Sebuah pelabelan dinamakan pelabelan* e-cordial *jika terdapat pemetaan biner* $f: E(G) \rightarrow \{0,1\}$ *yang menginduksi pelabelan simpul yang didefinisikan* $g(v) = \sum_{uv \epsilon E(G)} f(uv)(mod\ 2)$, *sehingga nilai mutlak dari selisih banyaknya simpul yang dilabeli 1 dan banyaknya simpul yang dilabeli* 0 *kurang dari sama dengan* 1, *dan nilai mutlak dari selisih banyaknya sisi yang dilabeli* 1 *dan banyaknya sisi yang dilabeli* 0 *kurang dari sama dengan* 1. *Sebuah graf yang dapat dilabeli secara* e-cordial *dinamakan graf* e-cordial. *Pada makalah ini dibuktikan bahwa beberapa graf kubah* $Cu(3, b, n)$ *adalah* e-cordial.
**Kata Kunci :** *Pelabelan* E-Cordial; *Graf* E-Cordial; *Graf Kubah* Cu(a, b, n).

## 1. INTRODUCTION

Graph labeling was first introduced in the mid-1960s [1]. The graph labeling can be applied to various sectors, including transportation systems, geographic navigation, radar, computer data storage, and radiofrequency transmitters. Especially in communication systems, graph labeling can reduce the risk of network topology vulnerabilities, where elements in the network are represented as vertices and connections between two elements are represented as edges. While in the computer data storage sector, anti-magic graph labeling can be applied as a data security strategy on computer servers.

One of many labeling techniques is e-cordial labeling. Yilmaz and Cahit [2] defined e-cordial labeling as a binary mapping $f: E(G) \rightarrow \{0,1\}$ that induces the labeling on the vertices defined by $g(v) = \sum_{uv \epsilon E(G)} f(uv)(mod2)$, so that the absolute value of the difference between the number of vertices labeled 1 and the number of vertices labeled 0 is less than or equal to 1, and also the absolute value of the difference between the number of edges labeled 0 and labeled 1 is at most 1.

Bala and Thirusangu [3] proved that the extended triplicate graph of finite paths of length $n$ where $n \equiv 1\ (mod\ 4)$ meets e-cordial and total e-cordial labelings. Vaidya and Bijukumar [4] showed that the split graph of cycle $C_n$ of even order, the joint sum of two copies of cycle $C_n$, and the shadowgraph

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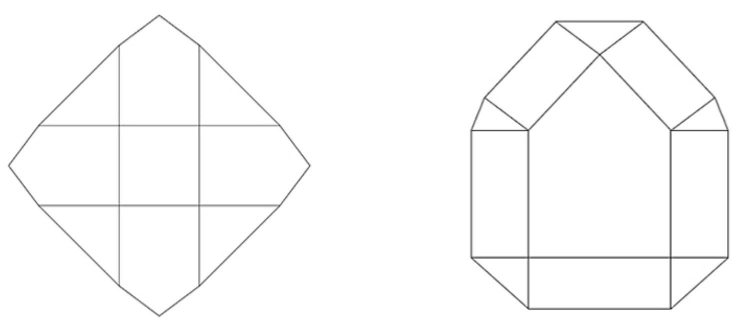
of path $P_n$ for even $n$ are e-cordial graphs. Vaidya and Vias [5] examined that closed helm $Ch_n$, flower graph $fl_n$, gear graph $G_n$, and double triangular snake $Dt_n$ are e-cordial graphs. Rahadjeng and Widyawati [6] proved that a complete graph, cycle graph, star graph, and wheel graph are e-cordial. In their research, they first determined the properties of the graphs, including the number of vertices, edges, and evenness of the number of edges and their cycles. Then, these properties are used to determine whether the graphs can be labeled e-cordially or not. Avudainayaki, Selvam, and Ulaganathan [7] proved that the extended duplicate graph of a splitting graph of path $P_m$, for $m \geq 2$ is e-cordial, product e-cordial, total e-cordial, and total product e-cordial. Bapat [8] showed that the bull graph consisting of a cycle $C_n$ and a pendant edge each at two adjacent vertices is e-cordial. In contrast to previous studies, in this paper, we propose a theorem for the existence of e-cordial labeling for some cupola graphs $Cu(a, b, n)$, for $a = 3$.

## 2. METHODS

In this section, based on some references, we provide definitions, properties of cupola graphs $Cu(a, b, n)$, and some previous results about e-cordial labeling. To derive an algorithm for e-cordial labeling for the cupola graphs $Cu(3, b, n)$, firstly, we define the notations of the vertices and the edges on the graph. Then we construct a function to label the edges such that the graph is e-cordial.

**Definition 1** [9]
Let $a, b$ and $n$ are integer $3 \leq a < b$ and $n \geq 3$. Cupola Graph $Cu(a, b, n)$ are a cycle graph $C_n$ surrounded by cycle graph $C_a$ and cycle graph $C_b$, where exactly one vertex $C_a$ intersects with every vertex in the cycle graph $C_n$, and exactly one edge $C_b$ intersects with every edge in the cycle graph $C_n$.

Examples of the cupola graph $Cu(3, 5, 4)$ and cupola graph $Cu(3, 4, 5)$ are shown in Fig1. Based on Definition 1, the number of vertices of the cupola graph $Cu(a, b, n)$ is $n(a + b - 4)$, and the number of edges on the cupola graph $Cu(a, b, n)$ is $n(a + b - 2)$.



(a) (b)

**Figure 1**. (a) The cupola graph $\mathbf{Cu(3, 5, 4)}$, (b) the cupola graph $\mathbf{Cu(3, 4, 5)}$.

**Definition 2** [10]
Given a graph $G$ with the vertex set $V(G)$ and the edge set $E(G)$. The function $f: E(G) \rightarrow \mathbb{Z} \cup 0$ is called edge labeling, function $g: V(G) \rightarrow \mathbb{Z} \cup 0$ is called vertex labeling, and $h: G = (E(G), V(G)) \rightarrow \mathbb{Z} \cup 0$ is called total labeling.

**Definition 3** [2]
Let $G = [V(G), E(G)]$ be a graph with two functions $f: E(G) \to \{0,1\}$ and $g(v) = \sum_{uv \in E(G)} f(uv) (mod\ 2)$.

The function $f$ is the e-cordial labeling of $G$ if it satisfies the following conditions:
1. $|e_f(0) - e_f(1)| \leq 1$,
2. $|v_g(0) - v_g(1)| \leq 1$,

where $e_f(0)$ and $e_f(1)$ represent the number of edges labeled 0 and 1, respectively, $v_g(0)$ and $v_g(1)$ represent the number of vertices labeled with 0 and 1. A graph that satisfies the e-cordial labeling is called an e-cordial graph.

**Theorem 1** [6]
If $G$ is an e-cordial graph with $n$ vertices then $n \not\equiv 2$ ($mod$ 4).

## 3. RESULTS AND DISCUSSIONS

Here, we prove the existence of e-cordial labeling for some cupola graphs $Cu(a,\ b,\ n)$, for $a = 3$.

**Theorem 2**
Let $G$ be a cupola graph $Cu(3, b, n)$, where $b > 3$ and $n \geq 3$. If $n(b-1) \equiv 0\ (mod\ 4)$ or $n(b-1) \equiv 1\ (mod\ 4)$ then $G$ is *e-cordial*.

**Proof.**
Firstly, we define the notation of the vertices and the edges on the cupola graph $Cu(3, b, n)$. These notations are used to define e-cordial labeling. The vertices of cycle graph $C_n$ are denoted by $\{v_1,\ v_2, \ldots, v_n\}$ sequentially in clockwise order. The edges of the cycle graph $C_n$ are sequentially denoted by $\{e_1,\ e_2, \ldots, e_n\}$, where $e_1 = v_n v_1$ and $e_i = v_{i-1} v_i$, for $i = 2, 3, .. n$. The outer vertices of the cupola graph are denoted by $\{v_{n+1},\ v_{n+2}, \ldots, v_{n(b-1)}\}$ sequentially in a clockwise order such that $v_{n+1}$ is adjacent with $v_1$. The edges that connect the vertices of the cycle graph $C_n$ with the outer vertices of the cupola graph are denoted by $\{e_{1,1},\ e_{1,2},\ e_{1,3}, \ldots, e_{1,2n}\}$ sequentially in a clockwise order where $e_{1,1} = v_1 v_{n+1}$. The outer edges of the cupola graph are denoted by $\{e_{2,1},\ e_{2,2},\ e_{2,3}, \ldots, e_{2,(b-2)n}\}$ sequentially in clockwise order, where $e_{2,1} = v_{n+1} v_{n+2}$.

Secondly, we label the edges by $f: E(G) \to \{0,1\}$ as follows: for labeling the edges of the cycle graph $C_n$, we define:

$$f(e_i) = \begin{cases} 0, \text{if } i \equiv 0, 3 (mod\ 4) \\ 1, \text{if } i \equiv 1, 2 (mod\ 4) \end{cases}, i = 1, 2, \ldots, n.$$

For the labeling of the edges connecting the cycle graph $C_n$ with the outer edges of the cupola graph, we define:

$$f(e_{1,j}) = \begin{cases} 0, \text{if } j \equiv 0\ (mod\ 2) \\ 1, \text{if } j \equiv 1\ (mod\ 2) \end{cases}, j = 1, 2, \ldots, 2n.$$

For labeling the outer edges of the cupola graph, we define two cases in cycle $b$, as follows:

Case 1: $b$ is even

$$f(e_{2,k}) = \begin{cases} 0, \text{if } k \equiv 0 \ (mod 2) \\ 1, \text{if } k \equiv 1 \ (mod 2) \end{cases}, k = 1,2, \dots, (b-2)n.$$

Case 2: $b$ is odd

$$f(e_{2,k}) = \begin{cases} 0, \text{if } k \equiv 1,2 \ (mod\ 4) \\ 1, \text{if } k \equiv 0,3 \ (mod\ 4) \end{cases}, k = 1,2, \dots, (b-2)n.$$

Finally, we give the label to the vertices of the cupola graph $Cu(3,b,n)$ by $g(v) = \sum_{uv \in E(G)} f(uv) (mod\ 2)$.

The defined labeling $f$ satisfies the vertex and edge conditions for e-cordial labeling, as shown in Table 1. Hence, the cupola graph $Cu(3,b,n)$ is e-cordial for $|V(G)| = n(b-1) \equiv 0\ (mod\ 4)$ or $|V(G)| = n(b-1) \equiv 1\ (mod\ 4)$.

**Table 1.** The numbers of vertices and edges are labeled 0 and 1.

| $\boldsymbol{\lvert V(G) \rvert}$ | **Edge condition** | **Vertex condition** |
|---|---|---|
| $\boldsymbol{n(b-1) \equiv 0\ (mod\ 4)}$ | $e_f(1) = e_f(0) = \frac{n(b+1)}{2}$ | $v_g(1) = v_g(0) = \frac{n(b-1)}{2}$ |
| $\boldsymbol{n(b-1) \equiv 1\ (mod\ 4)}$ | $e_f(1) - 1 = e_f(0) = \frac{n(b+1)-1}{2}$ | $v_g(1) + 1 = v_g(0) = \frac{n(b-1)-1}{2}$ |

For an illustration, let $a = 3, b = 4$, and $n = 3$. Firstly, the edges and vertices of the cupola graph $Cu(3,4,3)$ are named according to the rules in the proof section. The illustration is displayed in Figure 2. Secondly, we apply those functions $f$ to label the edges of the cupola graph $Cu(3,4,3)$. Finally, we label the vertices of the cupola graph with $g(v) = \sum_{uv \in E(G)} f(uv) (mod\ 2)$. The result of e-cordial labeling is displayed in Figure 3. It can be seen that $e_f(0) = 7$, $e_f(1) = 8$, $v_g(0) = 5$, and $v_g(1) = 4$, satisfying e-cordial labeling condition $|e_f(0) - e_f(1)| \leq 1$ and $|v_g(0) - v_g(1)| \leq 1$.

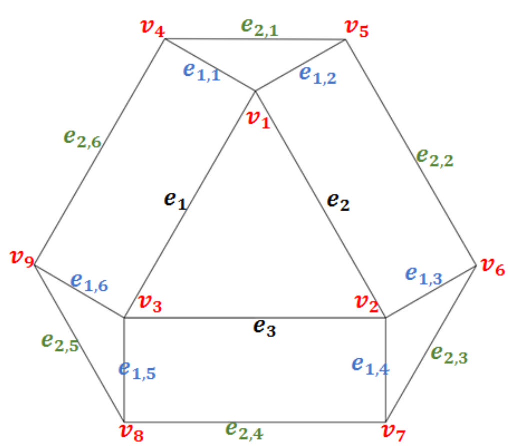



**Figure 2.** Notation for vertices and edges on cupola graph $\boldsymbol{Cu(3,4,3)}$.

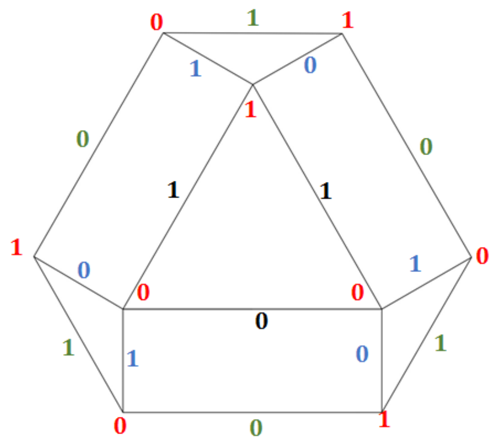



**Figure 3**. *E-cordial* labeling for cupola graph $\boldsymbol{Cu(3,4,3)}$.

## 4. CONCLUSIONS

We have shown that the cupola graphs $Cu(3,b,n)$ for $b > 3$ and $n \geq 3$ is e-cordial if the number of vertices of the cupola graph is congruent to $0\ (mod\ 4)$ or $1\ (mod\ 4)$. The possibility for e-cordial labeling of the cupola graphs $Cu(a,b,n)$ for $3 < a < b$ and $n \geq 3$ is an open area for future research.

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