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Numerical Results of Crank-Nicolson and Implicit Schemes to Laplace Equation with Uniform and Non-Uniform Grids

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Abstract

In this paper, we investigate the numerical results between Implicit and Crank-Nicolson method for Laplace equation. Based on the numerical results obtained, we get the conclusion that the absolute error of Crank-Nicolson method is smaller than the absolute error of Implicit method for uniform and non-uniform grids which both refer to the analytical solution of Laplace equation obtained by separable variable method.
Keywords: Crank-Nicolson; Implicit; Laplace equation; separable variable method; uniform and non-uniform grids.

Abstrak

Dalam makalah ini, kami menyelidiki hasil numerik antara etode Implisit dan Crank-Nicolson *untuk persamaan* Laplace. *Berdasarkan hasil numerik yang diperoleh, kita mendapatkan kesimpulan bahwa kesalahan absolut metode* Crank-Nicolson *lebih kecil daripada kesalahan absolut metode Implisit untuk grid seragam dan tak-seragam yang keduanya mengacu pada solusi analitik persamaan* Laplace *yang diperoleh dengan metode separable.*
Kata kunci: Crank-Nicolson*; Implisit; persamaan* Laplace*; metode variable terpisah; grid seragam dan tak-seragam.*

1. INTRODUCTION

All Mathematics was branch of science having the important role to solve the problem for many fields. The real phenomenon will be represented into the mathematical model that can be solved into analytical or numerical forms. In differential equations, mathematical models are divided into ordinary differential and partial differential equations.

In this research, the partial differential equations are studied to find the analytical solutions and numerical solutions. Many mathematical models are included into the category of partial differential equations. Laplace equation is one of this case where the second-order partial differential equation was discovered by a French mathematician and astronomer Pierre-Simon Laplace (1749-1827). In mathematics, Laplace equation is often written as $\Delta\phi = 0$ where Δ is Laplace operator or Laplacian and ϕ is a scalar function. Laplace equation has been widely used for many problems of electromagnetism, astronomy, and fluid dynamic, because they can be used to describe the behavior of electric, gravitational, and fluid potential. The analytical solution of Laplace equation can be established by applying the Separation Variable Method (SVM) and Fourier series. However, it needs much time to find the analytical solution with the boundary conditions given. For that reason, there are some numerical methods which can be used to solve numerically Laplace equation.

The last studies of numerical solutions for Laplace equation were studied by Marchi, etc. employing the Richardson extrapolation repeated for 2D Laplace equation [1], which is able to make

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the discretization error smaller. Rangogni [2] applied the boundary element method to the generalized Laplace equation. The study stated that the generalized Laplace equation can be overcame by employing the coupling boundary element method and perturbation method. Moreover, Rangogni and Occhi [3] establish the numerical results for the Laplace equation generalized using boundary element method. The procedure is applied to solve numerically three test problems of analytical solutions. Wei, etc. [4] applied new method of regularization according to the finite dimension of subspace. An appropriate choice was given to regularize the parameter and to give assumption of a-priori bound. A new a posteriori Fourier method was proposed to Laplace equation with nonhomogeneous Neumann which was studied by Fu, etc. [5]. Shojaei, etc. [6] studied the Laplace equation by using geometrical transformation and graph products.

Moreover, sometimes the numerical computations became very difficult to be established because there is no an a priori on the solution. It needs the regularization technique to get the stable approximation solution. Then, the various numerical methods were proposed to handle the ill-posedness problem, such as quasi-reversibility [7] [8] [9] [10], conjugate gradient [11], Tikhonov regularization [12] and finite difference method [13] [14], regularization of Lavrentiev [15], methods of moment [16] [11] [17], method of energy regularization [18].

Based on the last studies, we use Crank-Nicolson and Implicit method for Laplace equation, and using Thomas algorithm to establish the unknown values. The numerical results are then compared to analytical solution which is solved using separation of variable method. Moreover the rest of this paper consists of some Section. In Section 2, we discuss the discretization for the numerical method of Crank-Nicolson and Implicit. Then we apply the Thomas algorithm to find the unknown values. Section 3, we establish the numerical results and compare the results with the analytical solution. In this section, we use some various grid $N_x \times N_y$ which is not necessary same for those two grid of N_x and N_y.

2. DISCRETIZATION OF MATHEMATICAL MODEL

Given the following Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{1}$$

with the boundary conditions

$$\begin{aligned} u(x=0,y) &= 0, \quad u(x=1,y) = 0 \\ u(x,y=0) &= 0, \quad u(x,y=1) = f(x) = x(1-x)\sin(x) \end{aligned} \tag{2}$$

We further apply the Implicit method to (1), obtained

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{\Delta x^2} \tag{3}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{\Delta y^2} \tag{4}$$

where the positions for each u with respect to i and j can be shown the following stencil of Implicit method.

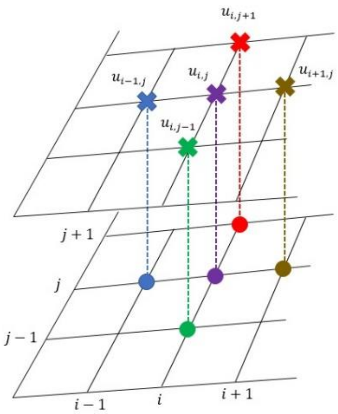


Figure 1. Stencil of Implicit method

Based on Figure 1, we use central difference for second derivate in space, where index i represents the axis-x and index j represents the axis-y. Moreover, $u(i,j)$ is the projection result at the axis-z of index (i,j) on the plane-xy. Then, by combining (3) and (4), we have

$$\left(u_{i-1,j} - 2u_{i,j} + u_{i+1,j}\right)r + \left(u_{i,j-1} - 2u_{i,j} + u_{i,j+1}\right) = 0 \tag{5}$$

where $r = \left(\frac{\Delta y}{\Delta x}\right)^2$.

Rearranging (5) to get

$$-ru_{i-1,j} - ru_{i+1,j} - u_{i,j-1} - u_{i,j+1} + 2(1+r)u_{i,j} = 0 \tag{6}$$

To establish the numerical solution of Implicit method for the Laplace equation, we first bring the discretization (6) into the tridiagonal matrix with index $i = 1,2,\ldots,N_x$ for axis-x and $j = 1,2,\ldots,N_y$ for axis-y. Based on the iteration result for discretization (6) and boundary conditions (2), we have the following block matrix system

$$\boldsymbol{AX} = \boldsymbol{B} \tag{7}$$

where

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{A_1} & \boldsymbol{C_1} & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \boldsymbol{D_1} & \boldsymbol{A_2} & \boldsymbol{C_2} & \cdots & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{D_2} & \ddots & \ddots & \boldsymbol{0} \\ \vdots & \vdots & \ddots & \boldsymbol{A_{n-1}} & \boldsymbol{C_n} \\ \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{D_n} & \boldsymbol{A_n} \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} \boldsymbol{B_1} \\ \boldsymbol{B_2} \\ \vdots \\ \boldsymbol{B_{n-1}} \\ \boldsymbol{B_n} \end{bmatrix}, \quad \boldsymbol{X} = \begin{bmatrix} \boldsymbol{X_1} \\ \boldsymbol{X_2} \\ \vdots \\ \boldsymbol{X_{n-1}} \\ \boldsymbol{X_n} \end{bmatrix} \tag{8}$$

$$\boldsymbol{A_i} = \begin{bmatrix} 2(1+r) & -r & 0 & \cdots & 0 \\ -r & 2(1+r) & -r & \cdots & 0 \\ 0 & -r & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & 2(1+r) & -r \\ 0 & 0 & \cdots & -r & 2(1+r) \end{bmatrix},$$

$$\boldsymbol{C_i} = \boldsymbol{D_i} = \begin{bmatrix} -1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & -1 & 0 \\ 0 & 0 & \cdots & 0 & -1 \end{bmatrix}$$

$$\boldsymbol{B_1} = \begin{bmatrix} ru_{0,1} + u_{1,0} \\ u_{2,0} \\ \vdots \\ u_{N_x-1,0} \\ ru_{N_x+1,1} + u_{N_x,0} \end{bmatrix}, \quad \boldsymbol{B_n} = \begin{bmatrix} ru_{0,N_y} + u_{1,N_y+1} \\ u_{2,N_y+1} \\ \vdots \\ u_{N_x-1,N_y+1} \\ ru_{N_x+1,N_y} + u_{N_x,N_y+1} \end{bmatrix}, \boldsymbol{B_q} = \begin{bmatrix} ru_{0,p} \\ 0 \\ \vdots \\ 0 \\ ru_{N_x+1,p} \end{bmatrix} \tag{9}$$

for $\boldsymbol{2 \leq p \leq N_y - 1}$ and $\boldsymbol{2 \leq q \leq n - 1}$.

To find the unknown values $\boldsymbol{X}$ in (7), we can use the Thomas algorithm. By using Gauss to eliminate the block matrix $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{X}$ in (8) first, one has

$$\begin{bmatrix} \boldsymbol{A_1^*} & \boldsymbol{C_1} & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{A_2^*} & \boldsymbol{C_2} & \cdots & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \ddots & \ddots & \boldsymbol{0} \\ \vdots & \vdots & \ddots & \boldsymbol{A_{n-1}^*} & \boldsymbol{C_n} \\ \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{0} & \boldsymbol{A_n^*} \end{bmatrix} \begin{bmatrix} \boldsymbol{X_1} \\ \boldsymbol{X_2} \\ \vdots \\ \boldsymbol{X_{n-1}} \\ \boldsymbol{X_n} \end{bmatrix} = \begin{bmatrix} \boldsymbol{B_1^*} \\ \boldsymbol{B_2^*} \\ \vdots \\ \boldsymbol{B_{n-1}^*} \\ \boldsymbol{B_n^*} \end{bmatrix} \tag{10}$$

where

$$\begin{aligned} \boldsymbol{A_1^*} &= \boldsymbol{A_1}, \ \boldsymbol{B_1^*} = \boldsymbol{B_1} \\ \boldsymbol{A_i^*} &= \boldsymbol{A_i} - \boldsymbol{C_{i-1}} \frac{\boldsymbol{D_i}}{\boldsymbol{A_{i-1}^*}} \\ \boldsymbol{B_i^*} &= \boldsymbol{B_i} - \boldsymbol{B_{i-1}^*} \frac{\boldsymbol{D_i}}{\boldsymbol{A_{i-1}^*}}, for\ \boldsymbol{i = 2, 3, \ldots, n-1, n} \end{aligned} \tag{11}$$

Then, by using backward substitution for $\boldsymbol{X}$, we have

$$\boldsymbol{X_n} = \frac{\boldsymbol{B_n^*}}{\boldsymbol{A_n^*}};\ \boldsymbol{X_k} = \frac{\boldsymbol{B_k^*} - \boldsymbol{C_k X_{k+1}}}{\boldsymbol{A_k^*}}, for\ \boldsymbol{k = n-1, n-2, \ldots, 2, 1} \tag{12}$$

Moreover, by applying the Crank-Nicolson method to (1), we get

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{2\Delta x^2} + \frac{u_{i-1,j+1} - 2u_{i,j+1} + u_{i+1,j+1}}{2\Delta x^2} \tag{13}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{2\Delta y^2} + \frac{u_{i+1,j-1} - 2u_{i+1,j} + u_{i+1,j+1}}{2\Delta y^2} \tag{14}$$

By combining (13) and (14), one has

$$\begin{aligned} &su_{i-1,j} + su_{i-1,j+1} + u_{i,j-1} + u_{i+1,j-1} - 2(s+1)u_{i,j} + (s-2)u_{i+1,j} + \\ &(1-2s)u_{i,j+1} + (s+1)u_{i+1,j+1} = 0 \end{aligned} \tag{15}$$

where $s = \frac{\Delta y^2}{\Delta x^2}$ and the positions for each u with respect to i and j can be shown the following stencil of Crank-Nicolson method.

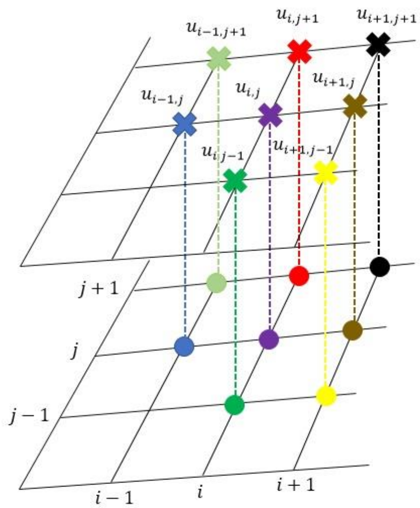


Figure 2. Stencil of Crank-Nicolson method

Based on Figure 2, we use average of central difference for second derivate in space, where index i represents the axis-x and index j represents the axis-y. Moreover, $u(i,j)$ is the projection result at the axis-z of index (i,j) on the plane-xy.

Similarly, to establish the numerical solution of Crank-Nicolson method for the Laplace equation, we first bring the discretization result (15) into the tridiagonal matrix with index $i = 1,2,\dots,N_x$ for axis-x and $j = 1,2,\dots,N_y$ for axis-y. Based on the iteration result for discretization (15) and boundary conditions (2), we have the following block matrix system

$$\boldsymbol{FX} = \boldsymbol{G} \tag{16}$$

where

$$\boldsymbol{F} = \begin{bmatrix} \boldsymbol{F_1} & \boldsymbol{P_1} & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \boldsymbol{Q_1} & \boldsymbol{F_2} & \boldsymbol{P_2} & \cdots & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{Q_2} & \ddots & \ddots & \boldsymbol{0} \\ \vdots & \vdots & \ddots & \boldsymbol{F_{n-1}} & \boldsymbol{P_n} \\ \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{Q_n} & \boldsymbol{F_n} \end{bmatrix}, \quad \boldsymbol{G} = \begin{bmatrix} \boldsymbol{G_1} \\ \boldsymbol{G_2} \\ \vdots \\ \boldsymbol{G_{n-1}} \\ \boldsymbol{G_n} \end{bmatrix}, \quad \boldsymbol{X} = \begin{bmatrix} \boldsymbol{X_1} \\ \boldsymbol{X_2} \\ \vdots \\ \boldsymbol{X_{n-1}} \\ \boldsymbol{X_n} \end{bmatrix} \tag{17}$$

$$\boldsymbol{F_i} = \begin{bmatrix} -2(1+s) & s-2 & 0 & \cdots & 0 \\ s & -2(1+s) & s-2 & \cdots & 0 \\ 0 & s & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & -2(1+s) & s-2 \\ 0 & 0 & \cdots & s & -2(1+s) \end{bmatrix}$$

$$\boldsymbol{P_i} = \begin{bmatrix} 1-2s & s+1 & 0 & \cdots & 0 \\ s & 1-2s & s+1 & \cdots & 0 \\ 0 & s & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & 1-2s & s+1 \\ 0 & 0 & \cdots & s & 1-2s \end{bmatrix}, \quad \boldsymbol{Q_i} = \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & \cdots & 0 \\ 0 & 0 & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & 1 & 1 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \tag{18}$$

$$\boldsymbol{G_1} = \begin{bmatrix} -su_{0,1} - su_{0,2} - u_{2,0} \\ -u_{2,0} - u_{3,0} \\ \vdots \\ -u_{N_x-1,0} - u_{N_x,0} \\ -(s-2)u_{N_x+1,1} - u_{N_x+1,0} - u_{N_x,0} - u_{N_x+1,0} \end{bmatrix}$$

$$\boldsymbol{G_n} = \begin{bmatrix} -su_{0,N_y} - u_{0,N_y+1} - (1-2s)u_{1,N_y+1} - (s+1)u_{2,N_y+1} \\ -su_{1,N_y+1} - (1-2s)u_{2,N_y+1} - (s+1)u_{3,N_y+1} \\ \vdots \\ -su_{N_x-2,N_y+1} - (1-2s)u_{N_x-1,N_y+1} - (s+1)u_{N_x,N_y+1} \\ -u_{N_x+1,N_y-1} - su_{N_x-1,N_y+1} - (s-2)u_{N_x+1,N_y} - (1-2s)u_{N_x,N_y+1} - (s+1)u_{N_x+1,N_y+1} \end{bmatrix}$$

$$\boldsymbol{G_q} = \begin{bmatrix} -su_{0,p} - su_{0,p+1} \\ 0 \\ \vdots \\ 0 \\ -u_{N_x+1,p-1} - (s-2)u_{N_x+1,p} - (s+1)u_{N_x+1,p+1} \end{bmatrix}$$

for $\boldsymbol{2 \le p \le N_y - 1}$ and $\boldsymbol{2 \le q \le n - 1}$.

To find the unknown values $\boldsymbol{X}$ in (16), we can use the Thomas algorithm. By using Gauss to eliminate the block matrix $\boldsymbol{F}, \boldsymbol{G}, \boldsymbol{X}$ in (17) first, one gets

$$\begin{bmatrix} \boldsymbol{F_1^*} & \boldsymbol{P_1} & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{F_2^*} & \boldsymbol{P_2} & \cdots & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \ddots & \ddots & \boldsymbol{0} \\ \vdots & \vdots & \ddots & \boldsymbol{F_{n-1}^*} & \boldsymbol{P_n} \\ \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{0} & \boldsymbol{F_n^*} \end{bmatrix} \begin{bmatrix} \boldsymbol{X_1} \\ \boldsymbol{X_2} \\ \vdots \\ \boldsymbol{X_{n-1}} \\ \boldsymbol{X_n} \end{bmatrix} = \begin{bmatrix} \boldsymbol{G_1^*} \\ \boldsymbol{G_2^*} \\ \vdots \\ \boldsymbol{G_{n-1}^*} \\ \boldsymbol{G_n^*} \end{bmatrix} \tag{19}$$

where

$$\boldsymbol{F_1^*} = \boldsymbol{F_1}, \;\; \boldsymbol{G_1^*} = \boldsymbol{G_1}$$

$$\boldsymbol{F_i^*} = \boldsymbol{F_i} - \boldsymbol{P_{i-1}} \frac{\boldsymbol{Q_i}}{\boldsymbol{F_{i-1}^*}} \tag{20}$$

$$\boldsymbol{G_i^*} = \boldsymbol{G_i} - \boldsymbol{G_{i-1}^*} \frac{\boldsymbol{Q_i}}{\boldsymbol{F_{i-1}^*}}, for\ \boldsymbol{i = 2, 3, \dots, n-1, n}$$

Then, by using backward substitution for $\boldsymbol{X}$, we have

$$\boldsymbol{X_n} = \frac{\boldsymbol{G_n^*}}{\boldsymbol{F_n^*}};\; \boldsymbol{X_k} = \frac{\boldsymbol{G_k^*} - \boldsymbol{P_k X_{k+1}}}{\boldsymbol{F_k^*}}, for\ \boldsymbol{k = n-1, n-2, \dots, 2, 1} \tag{21}$$

We further present the following algorithm which represents all the processes to get the numerical solutions.

Algorithm 1. Implicit method

Set: N_x, N_y and step size dx, dy
Determine:
$r = (dy/dx)^2$
% Tridiagonal block matrix $\boldsymbol{A}, \boldsymbol{C}, \boldsymbol{D}$
for $i = 1$ **to** N_X **do**
  for $j = 1$ **to** N_y **do**
    if $(i = j)$
    $\boldsymbol{A}(i,j) = 2 * (r + 1)$
    else
    $\boldsymbol{A}(i,j) = 0$
    end if
    if $(i = j + N_y - 2)$
    $\boldsymbol{C}(i,j) = -r$
    else
    $\boldsymbol{C}(i,j) = 0$
    end if
    if $(j = i + N_y - 2)$
    $\boldsymbol{D}(i,j) = -r$
    else
    $\boldsymbol{D}(i,j) = 0$
    end if
  end for
end for
% Boundary conditions B
for $i = 1$ **to** N_x **do**
  for $j = 1$ **to** N_y **do**
  $\boldsymbol{B}(i,j) = 0$; $\boldsymbol{B}(N_x, j) = 0$; $\boldsymbol{B}(i,1) = 0$; $\boldsymbol{B}(i, N_y) = x(i) - (1 - x(i))$
  end for
end for
% Final step
for $i = 1$ **to** N_x **do**
  for $j = 1$ **to** N_y **do**
  $\boldsymbol{A}^*(1,1) = \boldsymbol{A}(1,1)$; $\boldsymbol{B}^*(1,1) = \boldsymbol{B}(1,1)$
  $\boldsymbol{A}^*(i,j) = \boldsymbol{A}(i,j) - \boldsymbol{C}(i-1, j-1)\frac{\boldsymbol{D}(i,j)}{\boldsymbol{A}^*(i-1,j-1)}$
  $\boldsymbol{B}^*(i,j) = \boldsymbol{B}(i,j) - \boldsymbol{B}^*(i-1, j-1)\frac{\boldsymbol{D}(i,j)}{\boldsymbol{A}^*(i-1,j-1)}$
  end for
end for
for $i = N_x - 1$ **down to** 1 **do**
  for $j = N_y - 1$ **down to** 1 **do**
  $\boldsymbol{X}(N_x, N_y) = \frac{\boldsymbol{B}^*(N_x,N_y)}{\boldsymbol{A}^*(N_x,N_y)}$; $\boldsymbol{X}(i,j) = \frac{\boldsymbol{B}^*(i,j) - \boldsymbol{C}(i,j)\boldsymbol{X}(i+1,j+1)}{\boldsymbol{A}^*(i,j)}$
  end for
end for

Algorithm 2. Crank-Nicolson method

Set: N_x, N_y and step size dx, dy
Determine:
$r = (dy/dx)^2$
$\beta = -2 * (1 + 0.5 * ((dy/dx)^2))$
$\alpha = 0.5 * ((dy/dx)^2)$
% Tridiagonal block matrix $\boldsymbol{F}, \boldsymbol{P}, \boldsymbol{Q}$

```
for i = 1 to N_x do
    for j = 1 to N_y do
        if (i = j)
        F(i,j) = β
        else
        F(i,j) = 0
        end if
        if (i = j + N_y − 2)
        P(i,j) = α
        else
        P(i,j) = 0
        end if
        if (j = i + N_y − 2)
        Q(i,j) = α
        else
        Q(i,j) = 0
        end if
    end for
end for
% Boundary conditions G
for i = 1 to N_x do
    for j = 1 to N_y do
    G(i,j) = 0; G(N_x,j) = 0; G(i,1) = 0; G(i,N_y) = x(i) − (1 − x(i))
    end for
end for
% Final step
for i = 1 to N_x do
    for j = 1 to N_y do
    F*(1,1) = F(1,1);   G*(1,1) = G(1,1)
    F*(i,j) = F(i,j) − P(i − 1,j − 1) Q(i,j) / F*(i−1,j−1)
    G*(i,j) = G(i,j) − G*(i − 1,j − 1) Q(i,j) / F*(i−1,j−1)
    end for
end for
for i = N_x − 1 down to 1 do
    for j = N_y − 1 down to 1 do
    X(N_x,N_y) = G*(N_x,N_y) / F*(N_x,N_y);  X(i,j) = (G*(i,j)−P(i,j)X(i+1,j+1)) / F*(i,j)
    end for
end for
```

3. RESULTS AND DISCUSSIONS

We first present the following analytical solution obtained by separable variable method.

$$u(x,y) = \sum_{n=1}^{\infty} M_n \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi y}{a}\right), M_n = \frac{2}{a \sinh\left(\frac{n\pi}{a} b\right)} \int_0^a f\,(x) \sin\left(\frac{n\pi y}{a}\right) \quad (22)$$

(see Appendix for more detail of analytical solution by using SVM). Then, by using the boundary conditions (2) and the interval $(x,y) \in [0,a] \times [0,b] = [0,1] \times [0,1]$, (22) reduces to

$$u(x,y)=\sum_{n=1}^{\infty} M_n \frac{2\sin(n\pi x)\sinh(n\pi y)}{\sinh(n\pi)} \tag{23}$$

where

$$\begin{aligned} M_n &= -\frac{1}{2((n\pi-1)^3(n\pi+1)^3)}(\cos(\pi n-1)n^4\pi^4-\cos(\pi n+1)n^4\pi^4+2\cos(\pi n-1)n^3\pi^3) \\ &-\frac{1}{2((n\pi-1)^3(n\pi+1)^3)}(-2\sin(\pi n-1)n^3\pi^3+2\cos(\pi n+1)n^3\pi^3+2\sin(\pi n+1)n^3\pi^3) \\ &-\frac{1}{2((n\pi-1)^3(n\pi+1)^3)}(4n^3\pi^3-6\sin(\pi n-1)n^2\pi^2-6\sin(\pi n+1)n^2\pi^2) \\ &-\frac{1}{2((n\pi-1)^3(n\pi+1)^3)}(-2\cos(\pi n-1)n\pi-6\sin(\pi n-1)n\pi-2\cos(\pi n+1)n\pi) \\ &-\frac{1}{2((n\pi-1)^3(n\pi+1)^3)}(6\sin(\pi n+1)n\pi-4n\pi-\cos(\pi n-1)-2\sin(\pi n-1)) \\ &-\frac{1}{2((n\pi-1)^3(n\pi+1)^3)}(\cos(\pi n+1)-2\sin(\pi n+1)) \end{aligned}$$

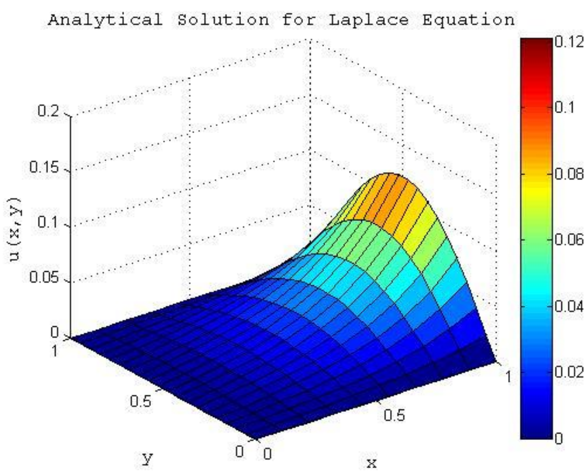
Then, we can establish the following numerical results and compare for each the numerical results of Implicit and Crank-Nicolson method with the analytical solution (22).

The simulation of analytical solutions (a) in Figure 3 to Figure 4 are obtained by using the iteration technique with grid $N_x \times N_y$, where this grid of $N_x \times N_y$ is not always necessary same. Moreover, the numerical results (b), (c) in Figure 3 to Figure 4 are obtained by using the finite difference method Implicit and Crank-Nicolson respectively. To do that, we first bring the discretization results into the block tridiagonal matrix. Moreover, to obtain the unknown values, we use the iteration technique of Thomas algorithm. Meanwhile, the absolute error (AE) values between Implicit and Crank-Nicolson (C-N) method which refer to analytical solution of Laplace equation are shown the table 1.

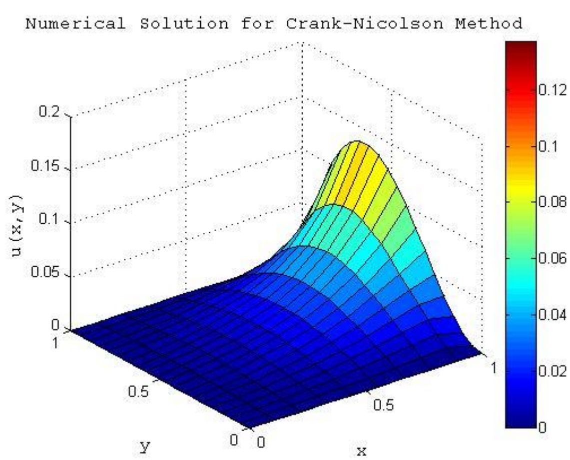
Based on the absolute error of Implicit and Crank-Nicolson on Table 1, we can conclude that the absolute error of Crank-Nicolson is always smaller than the absolute error of Implicit for all grids $N_x \times N_y$. Implicit method reaches the smallest absolute error of 0.0184 on the grid ($N_x = 20, N_y = 10$). Moreover, Crank-Nicolson method reaches the smallest absolute error of 0.0012 on the grid ($N_x = 20, N_y = 10$).

Table 1. Comparison of absolute error for Implicit and Crank-Nicolson

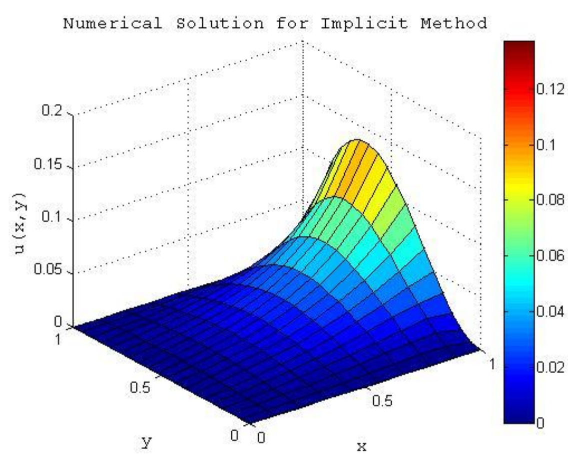
N_x	N_y	Analytical Solution at maximum iteration	Implicit at maximum iteration	C-N at maximum iteration	AE Implicit	AE C-N
20	10	0.5250	0.5434	0.5261	0.0184	0.0012
30	30	1.0397	1.0697	1.0577	0.0300	0.0180
30	40	1.1893	1.2225	1.2128	0.0332	0.0235
40	30	1.2058	1.2417	1.2265	0.0359	0.0207
40	40	1.3792	1.4191	1.4063	0.0398	0.0270



(a) Simulation of analytical solution $u(x, y)$

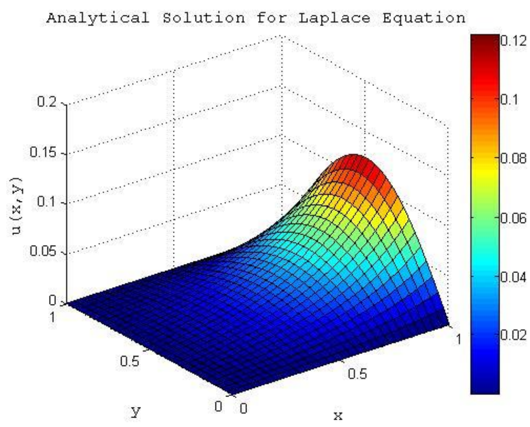


(b) Simulation of numerical solution (Crank-Nicolson method)

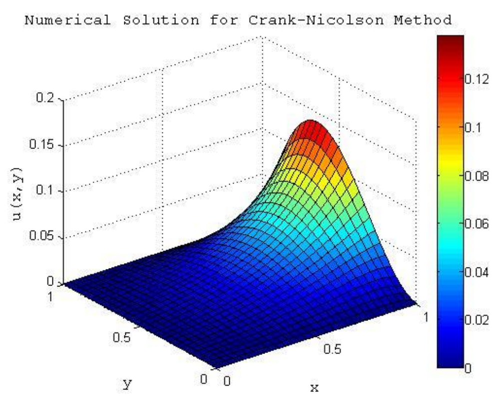


(c) Simulation of numerical solution (Implicit method)

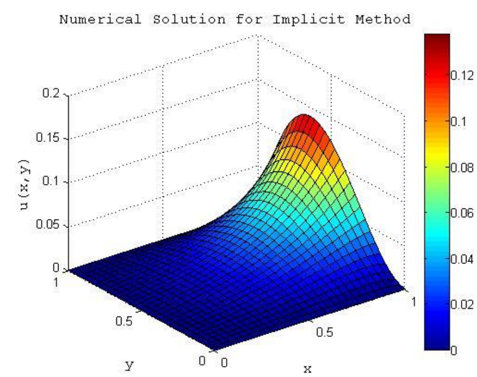
Figure 3. Simulation of analytical and numerical solutions on grid $[20 \times 10]$.



(a) Simulation of analytical solution $u(x,y)$



(b) Simulation of numerical solution (Crank-Nicolson method)



(c) Simulation of numerical solution (Implicit method)

Figure 4. Simulation of analytical and numerical solutions on grid $[30 \times 30]$

4. CONCLUSIONS

Based on the results in Table 1, Figure 3 to Figure 4, we can conclude that Crank-Nicolson is more stable than Implicit method, where these ones can be shown that the absolute error for each grid (uniform and non-uniform grids) of Crank-Nicolson is smaller than Implicit method. Moreover, it implies that Crank-Nicolson is more stable than Implicit method by achieving the smallest absolute error 0.0012 on the grid ($N_x = 20, N_y = 10$).

A. APPENDIX (ANALYTICAL SOLUTION OF LAPLACE EQUATION)

In this section, we will derive the analytical solution of Laplace equation by using Separation Variable Method (SVM). We first consider the boundary conditions as shown in (2), and present the following illustration in a diagram.

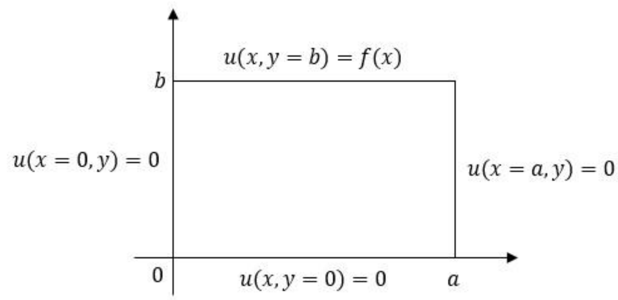


Figure 5. Diagram of the boundary conditions of Laplace equation

Based on the Figure 5, one has the interval $(x, y) \in [0, a] \times [0, b] = [0,1] \times [0,1]$. Then, we directly apply the SVM by substituting $u(x, y) = F(x)G(y)$ to get

$$\frac{\partial^2 F(x)}{\partial x^2}\frac{1}{F(x)} = -\frac{\partial^2 u}{\partial y^2}\frac{1}{G(y)} = -k \tag{A1}$$

which gives two ordinary differential equations

$$\frac{\partial^2 F(x)}{\partial x^2} + kF(x) = 0 \tag{A2}$$

and

$$\frac{\partial^2 G(y)}{\partial y^2} - kG(y) = 0 \tag{A3}$$

We solve ordinary differential equations of (A2) and (A3) respectively by the standard ways, then one has the following solutions:

(1) The solution of (A2)
Based on (A2), one has

$$F(x) = A\cos\sqrt{k}\,x + B\sin\sqrt{k}\,x$$

By applying the boundary conditions in Figure 5: $F(0) = 0 \Rightarrow A = 0$ and $F(a) = B\sin\sqrt{k}\,x = 0$ which implies that $k = \left(\frac{n\pi}{a}\right)^2, n \in \mathbb{Z}$. Then, we finally have the following solution of (A2)

$$F_n(x) = B_n \sin\left(\frac{n\pi}{a}x\right), n \in \mathbb{Z} \tag{A4}$$

(2) The solution of (A3)
Based on (A3) and $k = \left(\frac{n\pi}{a}\right)^2, n \in \mathbb{Z}$, one has

$$G_n(y) = A_n e^{\frac{n\pi}{a}y} + B_n e^{-\frac{n\pi}{a}y}, n \in \mathbb{Z}$$

By applying the boundary conditions in Figure 5: $G_n(0) = 0 \Rightarrow A_n = -B_n$ which gives

$$G_n(y) = A_n\left(e^{\frac{n\pi}{a}y} - e^{-\frac{n\pi}{a}y}\right), n \in \mathbb{Z}$$

Once again, by applying $e^{ay} - e^{-ay} = 2\sinh ay$, one has the following solution of (A3)

$$G_n(y) = A_n \sinh\left(\frac{n\pi}{a}y\right), n \in \mathbb{Z} \tag{A5}$$

We combine the solution $F_n(x)$ and $G_n(y)$, then one has

$$u(x,y) = \sum_{n=1}^{\infty} M_n \sinh\left(\frac{n\pi}{a}y\right) \sin\left(\frac{n\pi}{a}x\right) \tag{A6}$$

where $M_n = A_n \cdot B_n$. By substituting the boundary conditions at the top $u(x, y = b) = f(x)$ to (A6), one has

$$u(x,b) = \sum_{n=1}^{\infty} M_n \sinh\left(\frac{n\pi}{a}b\right) \sin\left(\frac{n\pi}{a}x\right) = f(x)$$

Since $M_n \sinh\left(\frac{n\pi}{a}b\right) = C_n$ is the constant, which gives

$$\sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{a}x\right) = f(x)$$

By employing the following Fourier series for C_n

$$C_n = \frac{2}{a}\int_0^a f(x) \sin\left(\frac{n\pi}{a}x\right) dx = M_n \sinh\left(\frac{n\pi}{a}b\right)$$

one has

$$M_n = \frac{2}{a \sinh\left(\frac{n\pi}{a}b\right)} \int_0^a f(x) \sin\left(\frac{n\pi}{a}x\right) dx \tag{A7}$$

Finally, we can derive the following general solution of Laplace equation by using Separation Variable Method

$$u(x,y) = \sum_{n=1}^{\infty} \left(\frac{2}{a \sinh\left(\frac{n\pi}{a}b\right)} \int_0^a f(x) \sin\left(\frac{n\pi}{a}x\right) dx \right) \sinh\left(\frac{n\pi}{a}y\right) \sin\left(\frac{n\pi}{a}x\right) \tag{A8}$$

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