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A Four-Parameter Extension of Burr III Distribution with Applications

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Abstract

In this paper, we defined and studied a new distribution called the odd exponentiated half-logistic Burr III distribution. Properties such as the linear representation of the probability density function (PDF) of the distribution, quantile function, ordinary and incomplete moments, moment generating function and distribution of the order statistic were derived. The PDF and hazard rate function were found to be capable of having various shapes, making the new distribution highly flexible. In particular, the hazard rate function can be nonincreasing, unimodal and nondecreasing. It can also have the bathtub shape among other non- monotone shapes. The maximum likelihood procedure was used to estimate the parameters of the new model. We gave two numerical examples to illustrate the usefulness and the ability of the distribution to provide better fits to a number of data sets than several distributions in existence.

Keywords: Burr III distribution; maximum likelihood procedure; moments; odd exponentiated half-logistic-G family; order statistics.

Abstrak

Pada artikel ini akan didefinisikan dan dipelajari mengenai distribusi baru yang disebut distribusi Burr III setengah logistik tereksponen ganjil. Kami menurunkan beberapa sifat dari distribusi tersebut yaitu representasi linier dari fungsi kepadatan peluang (FKP), fungsi kuantil, momen biasa dan momen tidak lengkap, fungsi pembangkit momen dan distribusi statistik terurut. Fungsi FKP dan fungsi tingkat hazard *diperoleh memiliki bermacam-macam bentuk, membuat distribusi baru ini sangat fleksibel. Secara khusus, fungsi tingkat* hazard *dapat berupa fungsi taknaik, bermodus tunggal, bisa juga tidak turun. Selain itu, fungsi ini juga dapat berbentuk seperti bak mandi di antara bentuk-bentuk tak monoton lainnya. Prosedur kemungkinan maksimum digunakan untuk mengestimasi parameter model yang baru. Kami memberikan dua contoh numerik untuk mengilustrasikan kegunaan dan kemampuan distribusi untuk menghasilkan kesesuaian yang lebih baik pada sejumlah kumpulan data dibandingkan beberapa distribusi yang ada.*

Kata kunci: *distribusi Burr III; prosedur kemungkinan maksimum; momen; keluarga setengah logistik-G teresponen ganjil; statistic terurut.*

1. INTRODUCTION

The Burr III (BIII) distribution, which is basically the distribution of the inverse transformation of the Burr XII random variable has found applications in actuarial science, environmental science, meteorology, reliability theory and survival analysis. The BIII distribution that depends on two parameters (a and b), where a and b are shape parameters, has the cumulative distribution function (CDF) and probability density function (PDF) defined by

$$G(x, a, b) = (1 + x^{-a})^{-b}, x > 0,\ a, b > 0, \tag{1}$$

and

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$$g(x,a,b)=abx^{-(a+1)}(1+x^{-a})^{-(b+1)}, x>0,\ a,b>0, \tag{2}$$

respectively.

Being one of the baseline distributions, there are situations the BIII distribution does not reasonably fit the data under consideration. In such situations, a generalization of the distribution can be considered. Several generalizations of the BIII distribution abound in the statistical science literature. [1] introduced the beta BIII distribution as well as the log-beta BIII regression model for analyzing censored data. The McDonald BIII distribution has been studied by [2], with emphasis on its mathematical properties and applications. A generalization of the BIII distribution called the modified BIII distribution has been introduced by [3]. In their paper, they showed categorically the relationships between the modified BIII distribution and each of the generalized inverse Weibull and loglogistic distributions. The transmuted and generalized BIII distributions were developed by [4] and [5], respectively. In another generalization of the BIII distribution, [6] introduced the odd BIII distributions. A special case of the gamma-generated family of distributions called the gamma BIII distribution was defined by [7]. Following the findings made by the authors, the hazard rate function of the distribution can be a decreasing, unimodal or decreasing-increasing –decreasing function. The log-gamma regression was also proposed by [7].

In this paper, we introduce and study a new extension of the BIII distribution called the odd exponentiated half logistic BIII (OEHLBIII) distribution, which can be sufficiently flexible to provide good fits to data from various fields. The new distribution is defined based on the odd exponentiated half logistic-G (OEHL-G) family of distributions introduced by [8].

Consider a parameter vector $\boldsymbol{\xi}$ and the corresponding baseline CDF $G(x,\boldsymbol{\xi})$. Let $g(x,\boldsymbol{\xi})$ be the baseline PDF. For $x\epsilon\mathbb{R}$ and two positive shape parameters α and λ, the CDF of the OEHL-G family has the form

$$F(x,\alpha,\lambda,\boldsymbol{\xi})=\left(\frac{1-\exp\left[\frac{-\lambda G(x,\xi)}{1-G(x,\xi)}\right]}{1+\exp\left[\frac{-\lambda G(x,\xi)}{1-G(x,\xi)}\right]}\right)^{\alpha}. \tag{3}$$

Associated with the CDF in (3) is the PDF

$$f(x,\alpha,\lambda,\boldsymbol{\xi})=2\alpha\lambda g(x,\boldsymbol{\xi})\frac{exp\left[\frac{-\lambda G(x,\xi)}{1-G(x,\xi)}\right]\left[1-exp\left[\frac{-\lambda G(x,\xi)}{1-G(x,\xi)}\right]\right]^{\alpha-1}}{[1-G(x,\xi)]^2\left[1+exp\left[\frac{-\lambda G(x,\xi)}{1-G(x,\xi)}\right]\right]^{\alpha+1}}. \tag{4}$$

Now, we proceed to determine the CDF and PDF of the OEHLBIII distribution. Substituting the CDF (1) into (3), the CDF of the OEHLBIII distribution is found to be

$$F(x,\alpha,\lambda,a,b)=\left(\frac{1-exp\left(\frac{\lambda}{1-(1+x^{-a})^{b}}\right)}{1+exp\left(\frac{\lambda}{1-(1+x^{-a})^{b}}\right)}\right)^{\alpha}, \alpha,\lambda,a,b>0, x>0. \tag{5}$$

By differentiating (5) with respect to x, we find that the OEHLBIII distribution has the PDF

$$f(x,\alpha,\lambda,a,b)=\frac{2\alpha\lambda abx^{-(a+1)}(1+x^{-a})^{-(b+1)}\,exp\left(\frac{\lambda}{1-(1+x^{-a})^{b}}\right)\left(1-exp\left(\frac{\lambda}{1-(1+x^{-a})^{b}}\right)\right)^{\alpha-1}}{\left(1-(1+x^{-a})^{-b}\right)^2\left(1+exp\left(\frac{\lambda}{1-(1+x^{-a})^{b}}\right)\right)^{\alpha+1}}. \tag{6}$$

In (5) and (6), the parameters α, λ, a and b are positive and shape parameters, making the OEHLBIII distribution highly flexible.

Next, we examine plots of the PDF and hazard rate function (HRF) of the distribution. The OEHLBIII PDF plots for some selected values of its parameters are presented in Figure 1.

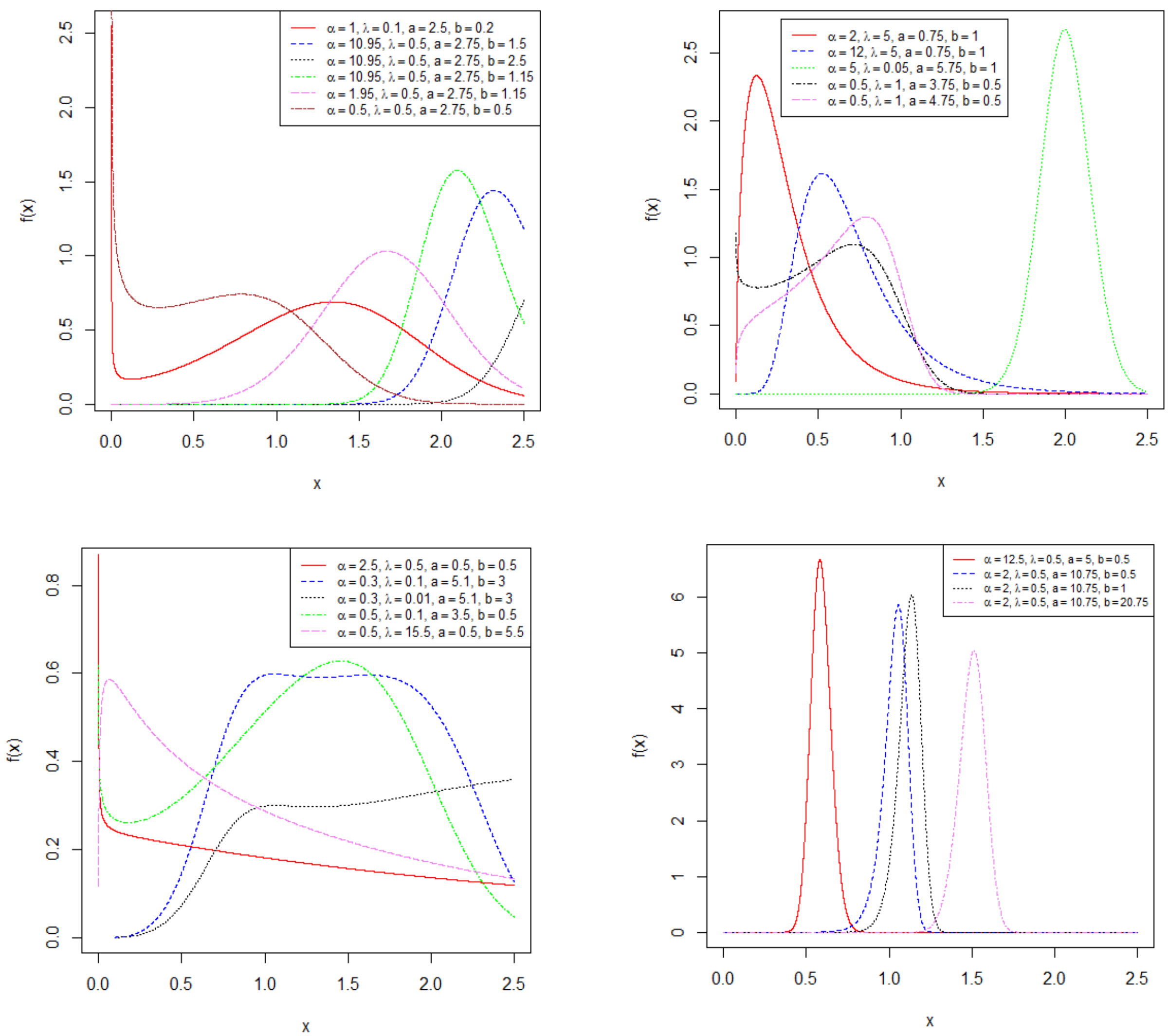

Figure 1. PDF of the OEHLBIII distribution for some selected parameter values.

The plots reveal that the PDF of the distribution can be left-skewed, right-skewed, nondecreasing, nonincreasing or unimodal. Given the OEHLBIII distribution, the hazard rate function (HRF) is defined to be

$$h(x) = \frac{f(x,\alpha,\lambda,a,b)}{1-F(x,\alpha,\lambda,a,b)} = \frac{f(x)}{1-F(x)}.$$

For the various shapes of the HRF, we consider Figure 2. In Figure 2, it is obvious that the HRF is capable of having any bathtub, upside down bathtub and L shapes. Additionally, the HRF can also be an increasing function or unimodal.

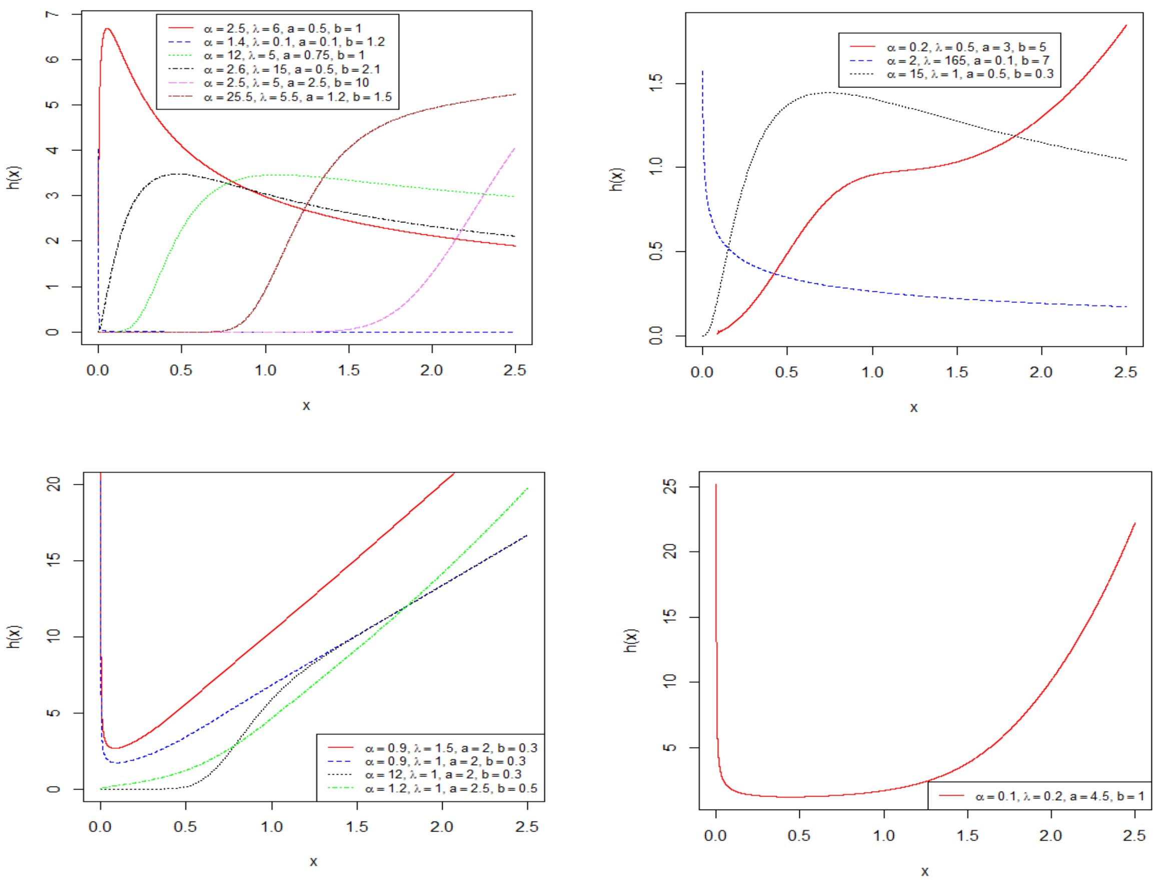

Figure 2. HRF of the OEHLBIII distribution for some selected parameter values.

2. PROPERTIES OF THE NEW DISTRIBUTION

In this section, we provide some mathematical properties of the new distribution.

2.1. Linear Representation of the OEHLBIII Distribution

The PDF (6) can be written as

$$f(x) = \sum_{k,l=0}^{\infty} a_{k,l}\, h_{k+l+1}(x), \tag{7}$$

such that $a_{k,1} = 2\alpha\lambda \sum_{i,j=0}^{\infty} \frac{(-1)^{j+k+l}(\lambda(i+j+k))^k}{k!(k+l+1)} \binom{-\alpha-1}{i}\binom{\alpha-1}{j}\binom{-k-2}{l}$ and $h_{k+l+1}(x) = (k+l+1)abx^{-(a+1)}(1+x^{-a})^{-b(k+l+1)-1}$ is the Burr III (BIII) density with power parameters a and b(k+l+1). With (7), it is possible to derive mathematical properties of the OEHLBIII distribution using those of the BIII distribution. Let Z be a BIII random variable. If $a > r$, the r-th raw moment and incomplete moment of Z are

$$\mu_r' = bB_2\left(1-\frac{r}{a}, b+\frac{r}{a}\right), \tag{8}$$

and

$$\varphi(t) = \int_0^t x^r\, g(x,a,b)dx = bB_2\left(t^{-\frac{1}{a}}, 1-\frac{r}{a}, b+\frac{r}{a}\right), \tag{9}$$

respectively, where

$$B_2(a,b) = \int_0^\infty z^{a-1}(z+1)^{-(a+b)}\,dz, \text{ and } B_2(t,a,b) = \int_t^\infty z^{a-1}(z+1)^{-(a+b)}\,dz,$$

are the beta and incomplete beta functions of the second kind.

2.2. Quantile Function and Random Number Generation

Suppose F(Q(w)) is the CDF of the OEHLBIII distribution evaluated at $x = Q(w)$. $Q(w)$ is called the quantile function for the distribution if $F(Q(w)) = w, 0 < w < 1$. Therefore

$$Q(w) = \left(\left(\frac{log_e\left(1-w^{\frac{1}{\alpha}}\right)-log_e\left(1+w^{\frac{1}{\alpha}}\right)-\lambda}{log_e\left(1-w^{\frac{1}{\alpha}}\right)-log_e\left(1+w^{\frac{1}{\alpha}}\right)}\right)^{\frac{1}{b}} - 1\right)^{-\frac{1}{a}}. \tag{10}$$

Let U denote a standard uniformly distributed variable. That is $U \sim U(0,1)$. By applying the inverse CDF technique, it can be shown that the variable

$$X = \left(\left(\frac{\log_e\left(1-U^{\frac{1}{\alpha}}\right)-\log_e\left(1+U^{\frac{1}{\alpha}}\right)-\lambda}{\log_e\left(1-U^{\frac{1}{\alpha}}\right)-\log_e\left(1+U^{\frac{1}{\alpha}}\right)}\right)^{\frac{1}{b}} - 1\right)^{-\frac{1}{a}}. \tag{11}$$

has the OEHLBIII distribution with parameters α, λ , a and b. In this regard, we write $X \sim OEHLBIII(\alpha, \lambda\,, a, b)$. Hence, for fixed values of α, λ , a and b, the OEHLBIII distributed data can be simulated using the formula

$$x = \left(\left(\frac{log_e\left(1-u^{\frac{1}{\alpha}}\right)-log_e\left(1+u^{\frac{1}{\alpha}}\right)-\lambda}{log_e\left(1-u^{\frac{1}{\alpha}}\right)-log_e\left(1+u^{\frac{1}{\alpha}}\right)}\right)^{\frac{1}{b}} - 1\right)^{-\frac{1}{a}}, \tag{12}$$

where $0 < u < 1$ and u is a random observation on U.

2.3. Raw and Incomplete Moments

For $a > r$ and with (8), the r-th raw moment of the OEHLBIII variable X is

$$\mu_r' = b\sum_{k,l=0}^{\infty} a_{k,l}(k+l+1)\beta_2\left(1-\frac{r}{a}, b(k+l+1)+\frac{r}{a}\right). \tag{13}$$

The mean of X corresponds to $r = 1$. The mean, variance, skewness and kurtosis of the distribution for various values of the parameters are shown in Table 1. Table 1 indicates that if α, a and b are fixed, the mean and variance of the OEHLBIII distribution decrease as λ increases. Additionally, the kurtosis is an increasing function of λ.

Table 1. Mean, Variance, Skewness and Kurtosis for Some Parameter Values of OEHLBIII Distribution

α	λ	a	b	Mean	Variance	Skewness	Kurtosis
0.5	0.5	0.5	0.5	1.4952	16.3649	7.3560	106.3337
0.5	1.5	0.5	0.5	0.1213	0.1407	8.5831	144.7057
0.5	2.5	0.5	0.5	0.0339	0.0133	9.7117	186.2283
0.5	3.5	0.5	0.5	0.0140	0.0026	10.4446	224.0405
0.5	5.0	0.5	0.5	0.0052	0.0005	12.1493	297.4591
0.5	0.5	1.5	2.0	2.1803	2.7013	1.0580	4.1035
0.5	1.5	1.5	2.0	1.1628	0.6470	0.9569	3.8894
0.5	2.5	1.5	2.0	0.8879	0.3432	0.8784	3.7117
0.5	3.5	1.5	2.0	0.7496	0.2296	0.8187	3.5722
0.5	5.0	1.5	2.0	0.6309	0.1524	0.7499	3.4090
0.5	0.5	1.5	0.5	0.6699	0.4270	1.2053	2.7289
1.5	0.5	1.5	0.5	1.2416	0.4471	0.6626	3.5325
2.5	0.5	1.5	0.5	1.5187	0.4067	0.6161	3.6295
3.5	0.5	1.5	0.5	1.6947	0.3776	0.6253	3.7253
5.0	0.5	1.5	0.5	1.8704	0.3625	0.4615	4.2683
1.5	2.0	0.5	2.5	10.1153	161.8494	3.8169	31.5417
1.5	2.0	1.5	2.5	1.8939	0.5400	0.6395	3.6150
1.5	2.0	2.5	2.5	1.4401	0.1164	0.1752	2.9700
1.5	2.0	3.5	2.5	1.2900	0.0487	-0.0054	2.6650
1.5	2.0	5.0	2.5	1.1913	0.0209	-0.2309	3.4431
1.5	2.5	2.0	0.5	0.4167	0.0444	0.4697	3.0577
1.5	2.5	2.0	1.5	1.0504	0.1138	0.3016	3.1239
1.5	2.5	2.0	2.5	1.4570	0.1732	0.3340	3.1591
1.5	2.5	2.0	3.5	1.7769	0.2313	0.3598	3.1762
1.5	2.5	2.0	5.0	2.1719	0.3185	0.3800	3.1950

If λ, a and b are kept constant, the mean increases as α increases. Holding α, λ and b constant results in the decreasing values of the mean, variance, skewness and kurtosis as a increases. Mean, variance and kurtosis increase as b increases provided the other parameters are constant. Using (7) and (9), the r-th incomplete moment of the distribution is found to be

$$\phi_r(t) = b\sum_{k,l=0}^{\infty} a_{k,l}(k+l+1)\beta_2\left(t^{-\frac{1}{a}}, 1-\frac{r}{a}, b(k+l+1)+\frac{r}{a}\right). \tag{14}$$

2.4. Moment Generating Function

We can express the MGF of the OEHLBIII distribution as

$$M_X(t) = \sum_{k,l=0}^{\infty} a_{k,l}\, M_{b(k+l+1)}(t), \tag{15}$$

where $M_{b(k+l+1)}(t)$ is the MGF of the BIII distribution with parameters a and $b(k+l+1)$. [1] have derived the MGF of a three-parameter BIII distribution with two shape parameters α and β and a scale parameter s, leading to the formula

$$M_{BIII}(t) = \frac{\beta s m}{p} I'\left(-st, \frac{\beta m}{p}-1, \frac{m}{p}, -\beta-1\right), t<0, \tag{16}$$

where $\alpha = \frac{m}{p}$, such that both m and p are positive integers. Next, we consider a special case of (16) in which $\alpha = a, \beta = b$ and $s = 1$to obtain

$$M_{b(k+l+1)}(t) = abI'(-t, ab-1, a, -b-1). \tag{17}$$

Therefore $M_X(t) = ab \sum_{k,l=0}^{\infty} a_{k,l}\, I'(-t, ab-1, a, -b-1)$, where

$$I'(-t, ab-1, a, -b-1) = \int_0^{\infty} x^{ab-1}(1+x^a)^{-b-1}\, exp(tx)\, dx.$$

2.5. Order Statistics

Suppose we have a random sample $X_1, X_2, X_3, \dots, X_n$ of size n from the OEHLBIII distribution and the corresponding order statistics $X_{(1)}, X_{(2)}, X_{(3)}, \dots, X_{(n)}$. The pdf of the i-th order statistic can be written as [9]

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-1)!} f(x) \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} F^{i+j-1}(x). \tag{18}$$

By applying Equation (20) in [8], we have

$$f(x)F^{i+j-1}(x) = \sum_{s,w,k,l=0}^{\infty} \frac{2\alpha\lambda^{k+1}ab(-1)^{s+k+l}x^{-(a+1)}(1+x^{-a})^{-b(k+2)-1}}{k!(s+w+1)^{-k}} \binom{\alpha(i+j)-1}{s}$$
$$\times \binom{-\alpha(i+j)-1}{w} \binom{-k-2}{l} \quad . \tag{19}$$

Substituting (19) into (18) leads to

$$f_{i:n}(x) = \sum_{k,l=0}^{\infty} b_{k,l}\, h_{b(k+l+1)}, \tag{20}$$

where $h_{b(k+l+1)}$ refers to the BIII density with parameters a and $b(k+l+1)$ and

$$b_{k,l} = \sum_{j=0}^{n-i} \sum_{s,w=0}^{\infty} \frac{2\alpha\lambda^{k+1}ab(-1)^{j+k+l+s}}{k!\,(k+l+1)(s+w+1)^{-k}} \binom{\alpha(i+j)-1}{s} \binom{-\alpha(i+j)-1}{w} \binom{-k-2}{l}.$$

Furthermore, the r-th moment of the i-th order statistic is

$$E({X_{i:n}}^r) = b \sum_{k,l=0}^{\infty} b_{k,l}\, (k+l+1)\beta_2\left(1-\frac{r}{a}, b(k+l+1)+\frac{r}{a}\right). \tag{21}$$

3. ESTIMATION

The maximum likelihood estimation of the parameters of the OEHLBIII distribution is implemented by maximizing the associated likelihood function. For a random sample of size n from the OEHLBIII distribution, the log-likelihood function is

$$log\, L = n\, log(2\alpha\lambda ab) - (a+1)\sum_{i=1}^{n} log\, x_i - (b+1)\sum_{i=1}^{n} log(1+{x_i}^{-a}) + \lambda \sum_{i=1}^{n} t_i$$
$$+(\alpha-1)\sum_{i=1}^{n} log(1-exp(\lambda t_i)) - 2\sum_{i=1}^{n} log\left(1-(1+{x_i}^{-a})^{-b}\right) - (\alpha+1) \times$$
$$\sum_{i=1}^{n} log(1+exp(\lambda t_i)),$$

where $t_i = \left(1 - (1 + x_i^{-a})^b\right)^{-1}$.

The partial derivatives associated with $log\, L$ are

$$\frac{\partial\, log\, L}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} log(1 - exp(\lambda t_i)) - \sum_{i=1}^{n} log(1 + exp(\lambda t_i)),$$

$$\frac{\partial\, log\, L}{\partial \lambda} = \frac{n}{\lambda} + \sum_{i=1}^{n} t_i - (\alpha - 1)\sum_{i=1}^{n} \frac{t_i\, exp(\lambda t_i)}{1 - exp(\lambda t_i)} - (\alpha + 1)\sum_{i=1}^{n} \frac{t_i\, exp(\lambda t_i)}{1 + exp(\lambda t_i)},$$

$$\frac{\partial\, log\, L}{\partial a} = \frac{n}{a} - \sum_{i=1}^{n} log\, x_i + (b+1)\sum_{i=1}^{n} \frac{x_i^{-a}\, log\, x_i}{1 + x_i^{-a}} - \lambda b \sum_{i=1}^{n} x_i^{-a}(1 + x_i^{-a})^{b-1} t_i^2\, log\, x_i$$

$$+\lambda b(\alpha - 1)\sum_{i=1}^{n} \frac{x_i^{-a}(1 + x_i^{-a})^{b-1} t_i^2\, exp(\lambda t_i)\, log\, x_i}{1 - exp(\lambda t_i)}$$

$$+2b\sum_{i=1}^{n} \frac{x_i^{-a}\, log\, x_i}{(1 - (1 + x_i^{-a})^{-b})(1 + x_i^{-a})^{b+1}}$$

$$+\lambda b(\alpha + 1)\sum_{i=1}^{n} \frac{x_i^{-a}(1 + x_i^{-a})^{b-1} t_i^2\, exp(\lambda t_i)\, log\, x_i}{1 + exp(\lambda t_i)}$$

and

$$\frac{\partial\, log\, L}{\partial b} = \frac{n}{b} - \sum_{i=1}^{n} log(1 + x_i^{-a}) + \lambda \sum_{i=1}^{n} (1 + x_i^{-a})^b t_i^2\, log(1 + x_i^{-a})$$

$$-\lambda(\alpha - 1)\sum_{i=1}^{n} \frac{(1 + x_i^{-a})^b t_i^2\, exp(\lambda t_i)\, log(1 + x_i^{-a})}{1 - exp(\lambda t_i)}$$

$$-2\sum_{i=1}^{n} \frac{(1 + x_i^{-a})^{-b}\, log(1 + x_i^{-a})}{1 - (1 + x_i^{-a})^{-b}} - \lambda(\alpha + 1)\sum_{i=1}^{n} \frac{(1 + x_i^{-a})^b t_i^2\, exp(\lambda t_i)\, log(1 + x_i^{-a})}{1 + exp(\lambda t_i)}.$$

Finding the maximum likelihood estimates of the respective parameters amounts to solving the equations $\frac{\partial\, log\, L}{\partial \alpha} = 0$, $\frac{\partial\, log\, L}{\partial \lambda} = 0$, $\frac{\partial\, log\, L}{\partial a} = 0$ and $\frac{\partial\, log\, L}{\partial b} = 0$ simultaneously. Since the analytical solution of the equations cannot be found, a numerical approach to solving the equations may be considered.

4. APPLICATIONS

In this section, we illustrate the flexibility and applicability of OEHLBIII distribution (OEHLBIIID) using two real data sets. The first data (Data 1) comprising the annual maximum daily precipitation data (in millimeters) which was reported in Busan, Korea, from 1904 to 2011 are recorded in Table 2. Data 1 have been modelled by authors such as [10] [11] [12] [13]. The second data set (Data 2) was reported by [14] and subsequently modelled by [3]. The data (fracture toughness MPa $m^{1/2}$ data from the material Alumina) are presented in Table 3.

Table 2. Annual maximum daily precipitation data

24.8,140.9,54.1,153.5,47.9,165.5,68.5,153.1,254.7,175.3,87.6,150.6,147.9,354.7,128.5,150.4,
119.2,69.7,185.1,153.4,121.7,99.3,126.9,150.1,149.1,143,125.2,97.2,179.3,125.8,101,89.8,
54.6,283.9,94.3,165.4,48.3,69.2,147.1,114.2,159.4,114.9,58.5,76.6,20.7,107.1,244.5,126,122.2,219.9,153.2,145.3,101.9,13
5.3,103.1,74.7,174,126,144.9,226.3,96.2,149.3,122.3,164.8,188.6,273.2,61.2,84.3,130.5,96.2,155.8,194.6,92,131,137,106.
8,131.6,268.2,124.5,147.8,294.6,
101.6,103.1,247.5,140.2,153.3,91.8,79.4,149.2,168.6,127.7,332.8,261.6,122.9,273.4,178,177,
108.5,115,241,76,127.5,190,259.5,301.5.

Table 3. Fracture toughness data

5.5, 5, 4.9, 6.4, 5.1, 5.2, 5.2, 5, 4.7, 4, 4.5, 4.2, 4.1, 4.56, 5.01, 4.7, 3.13, 3.12, 2.68, 2.77, 2.7, 2.36, 4.38, 5.73, 4.35, 6.81, 1.91,
2.66, 2.61, 1.68, 2.04, 2.08, 2.13, 3.8, 3.73, 3.71, 3.28, 3.9, 4, 3.8, 4.1, 3.9, 4.05, 4, 3.95, 4, 4.5, 4.5, 4.2, 4.55, 4.65, 4.1, 4.25,
4.3, 4.5, 4.7, 5.15, 4.3, 4.5, 4.9, 5, 5.35, 5.15, 5.25, 5.8, 5.85, 5.9, 5.75, 6.25, 6.05, 5.9, 3.6, 4.1, 4.5, 5.3, 4.85, 5.3, 5.45, 5.1,
5.3, 5.2, 5.3, 5.25, 4.75, 4.5, 4.2, 4, 4.15, 4.25, 4.3, 3.75, 3.95, 3.51, 4.13, 5.4, 5, 2.1, 4.6, 3.2, 2.5, 4.1, 3.5, 3.2, 3.3, 4.6, 4.3,
4.3, 4.5, 5.5, 4.6, 4.9, 4.3, 3, 3.4, 3.7, 4.4, 4.9, 4.9, 5

For the two data, we compare the fits of OEHLBIID with those of beta Burr III distribution (BBIID) [1], Burr III distribution (BIIID), gamma Burr III distribution (GBIIID) [7], Kumaraswamy Burr III distribution (KBIIID) [15] and modified Burr III distribution (MBIIID) [3]. Notably, the PDFs associated with BBIID, GBIIID, KBIIID and MBIIID are respectively given by

$$f(\alpha,\lambda,a,b,s)=\frac{ab}{s\left(\frac{x}{s}\right)^{a+1}B(\alpha,\lambda)}\left[\frac{\left(\frac{x}{s}\right)^{a}}{1+\left(\frac{x}{s}\right)^{a}}\right]^{\alpha b+1}\left[1-\left(\frac{\left(\frac{x}{s}\right)^{a}}{1+\left(\frac{x}{s}\right)^{a}}\right)^{b}\right]^{\lambda-1},\alpha,\lambda,a,b,s>0,x>0;$$

$$f(\alpha,a,b,s)=\frac{ab}{s\left(\frac{x}{s}\right)^{a+1}\Gamma(\alpha)}\left[\frac{\left(\frac{x}{s}\right)^{a}}{1+\left(\frac{x}{s}\right)^{a}}\right]^{b+1}\left[-log\left(1-\left(\frac{\left(\frac{x}{s}\right)^{a}}{1+\left(\frac{x}{s}\right)^{a}}\right)^{b}\right)\right]^{\alpha-1},\alpha,a,b,s>0,x>0;$$

$$f(\alpha,\lambda,a,b)=\alpha\lambda abx^{-(a+1)}(1+x^{-a})^{-(\alpha b+1)}\left(1-(1+x^{-a})^{-\alpha b}\right)^{\lambda-1},\alpha,\lambda,a,b>0,x>0;$$

$$f(\alpha,a,b)=abx^{-(a+1)}(1+\alpha x^{-a})^{-\left(\frac{b}{\alpha}+1\right)},\alpha,a,b>0,x>0.$$

Notably, all the numerical results in this section are obtained using the fitdistrplus package in R. The optimization of the log-likelihood function associated with each of the six multi-parameter distributions is carried out using the default method for distributions with more than one parameter (Nelder-Mead method).

To compare the goodness of fits of the six models, we use the Akaike Information Criteria (AIC), Bayesian Information Criteria (BIC), Kolmogorov-Smirnov Statistic (KS), Cramer-von Mises (W^*) and Anderson-Darling Statistic (A^*). The distribution with the best fit to each data is the distribution corresponding to minimum values of AIC, BIC, KS, W^* and A^*.

Table 4 contains maximum likelihood estimates (MLEs) of the parameters of the distributions fitted to Data 1 and the corresponding values of AIC, BIC, KS, W^* and A^*. The results presented in Table 4 show that the OEHLBIIID has the lowest value of AIC, BIC, KS, W^* and A^*. Thus, the OEHLBIIID is the best among the six models fitted to Data 1.

Table 4. MLEs of the models for Data 1, the associated standard error estimates and the values of AIC, BIC, KS, W^* and A^*.

Models	Estimates	Standard Error Estimates	-log L	AIC	BIC	KS	W^*	A^*
BBIII	$\hat{\alpha}$=9.6204	0.0034	723.5806	1457.161	1470.431	0.9590	30.8685	234.7095
	$\hat{\lambda}$=0.8590	0.0092						
	$\hat{a}$=1.2557	0.4417						
	$\hat{b}$=0.0405	0.0040						
	$\hat{s}$=7.8582	0.0046						
BIII	$\hat{a}$=1.6501	0.1028	608.5529	1221.106	1226.414	0.1453	0.8947	5.3741
	$\hat{b}$=1992.9226	893.2800						
GBIII	$\hat{\alpha}$=21.1858	0.4501	594.8563	1197.713	1208.328	0.1300	0.5158	2.9587
	$\hat{a}$=7.8053	0.0230						
	$\hat{b}$=18.6784	0.1153						
	$\hat{s}$=5.9333	0.0175						
KBIII	$\hat{\alpha}$=0.8890	0.0103	783.1645	1574.329	1584.945	.5359	8.7151	40.2025
	$\hat{\lambda}$=0.0332	0.0032						
	$\hat{a}$=6.1611	0.0034						
	$\hat{b}$=0.6665	0.0110						
MBIII	$\hat{\alpha}$=48.0216	74.6224	607.2952	1220.59	1228.552	0.1388	0.7964	4.8213
	$\hat{a}$=1.7387	0.1329						
	$\hat{b}$=3062.9954	1766.040 2						
OEHL BIII	$\hat{\alpha}$=2.9189	1.3915	580.8811	1169.762	1180.378	0.0905	0.1253	0.7427
	$\hat{\lambda}$=0.0041	0.0006						
	$\hat{a}$=1.1794	0.2566						
	$\hat{b}$=0.6322	0.9002						

Figure 3 shows the histogram, estimated densities and estimated CDFs for Data 1. Based on this figure, we infer that the OEHLBIIID is suitable for Data 1.

In Table 5, we have the MLEs of the parameters of the models fitted to Data 2, the corresponding standard errors and AIC, BIC, KS, W^* and A^* values. On the basis of lowest AIC, BIC, KS, W^* and A^* values, the OEHLBIID is the most suitable model among all the models fitted to the data.

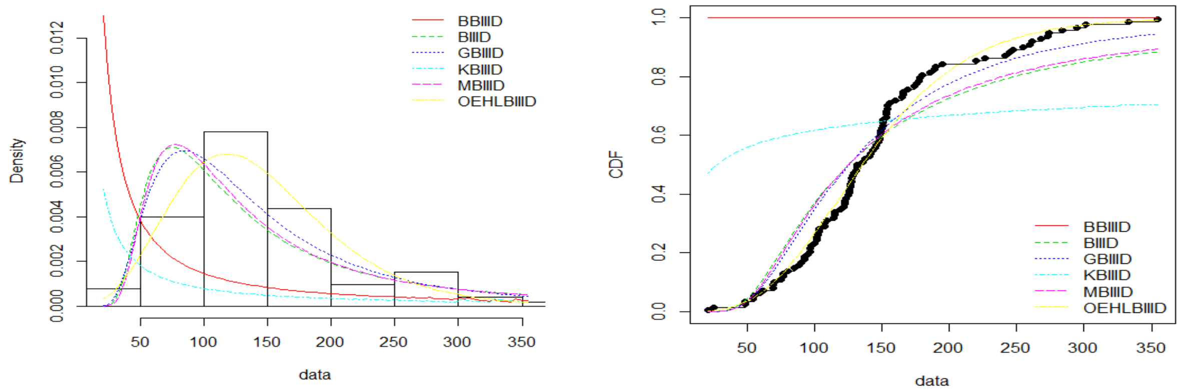

Figure 3: Estimated PDFs (left panel) and CDFs (right panel) for Data 1

Table 5. MLEs of the parameters of the models for Data 2, the associated standard error estimates and the values of AIC, BIC, KS, W* and A*

Models	Estimates	Standard Error Estimate	-log L	AIC	BIC	KS	W*	A*
BBIII	$\hat{\alpha}$=0.0561	0.1786	167.8023	345.6047	359.5003	0.9426	33.2240	228.0259
	$\hat{\lambda}$=1.3122	1.2550						
	$\hat{a}$=12.8125	5.6904						
	$\hat{b}$=5.7520	18.2532						
	$\hat{s}$=5.2035	0.2050						
BIII	$\hat{a}$=3.0607	0.1802	209.7675	423.5350	429.0932	0.1964	1.4297	8.1098
	$\hat{b}$=52.0622	11.2352						
GBIII	$\hat{\alpha}$=0.2000	0.0182	198.2675	404.5350	415.6515	0.7898	23.1824	120.2560
	$\hat{a}$=5.1503	0.0027						
	$\hat{b}$=9.3326	0.0070						
	$\hat{s}$=3.5632	0.0027						
KBIII	$\hat{\alpha}$=1191.205	259.9667	173.9927	355.9854	367.1019	0.1114	0.2716	1.6938
	$\hat{\lambda}$=120.4422	80.3170						
	$\hat{a}$=1.0101	0.1176						
	$\hat{b}$=0.0209	0.0042						
MBIII	$\hat{\alpha}$=1201.3094	760.4504	185.5963	377.1927	385.5301	0.1438	0.6534	3.8825
	$\hat{a}$=5.0924	0.3081						
	$\hat{b}$=1433.1242	673.0790						
OEHL BIII	$\hat{\alpha}$=0.9422	0.2967	167.6595	343.3191	354.4356	0.0674	0.0731	0.4411
	$\hat{\lambda}$=0.0220	0.0220						
	$\hat{a}$=4.3389	0.8168						
	$\hat{b}$=12.9473	20.9565						

Also, Figure 4 reveals that the OEHLBIID is a good model for the data.

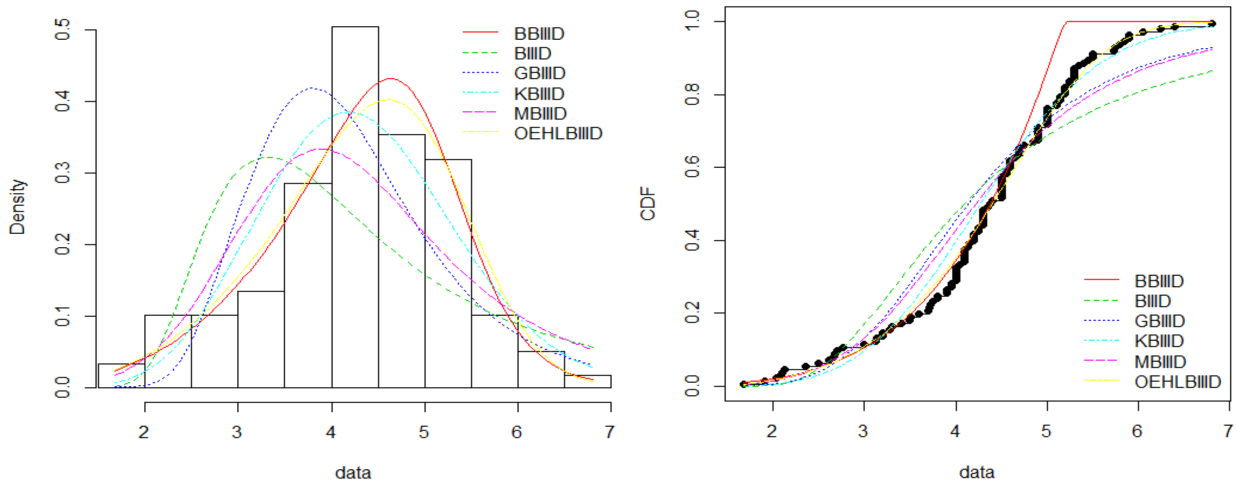

Figure 4. Estimated PDFs (left panel) and CDFs (right panel) for Data 2

5. CONCLUSIONS

We have extended the two-parameter Burr III distribution to obtain a new distribution called the odd exponentiated half-logistic Burr III distribution. The new distribution can be applied in reliability analysis, survival analysis, time series analysis among other fields. Properties of the distribution, namely, the linear representation of its density function, quantile function, raw and incomplete moments, moment generating function and distribution of the order statistic have been determined. The maximum likelihood method of estimating the parameters of the distribution was discussed. Comparatively speaking, the PDF and hazard rate function of the distribution introduced in this article are capable of having shapes that the PDF and hazard rate function of the baseline distribution do not have. Hence, the new model is more flexible than its corresponding baseline distribution. The numerical results obtained in this study indicate that the new distribution can be a better distribution for several data sets than many well-known continuous distributions, especially its sub model the two-parameter Burr III distribution.

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