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On Triangular Secure Domination Number

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Abstract

Let $T_m = (V(T_m), E(T_m))$ be a triangular grid graph of $m \in \mathbb{N}$ level. The order of graph T_m is called a triangular number. A subset T of $V(T_m)$ is a dominating set of T_m if for all $u \in V(T_m)\setminus T$, there exists $v \in T$ such that $uv \in E(T_m)$, that is, $N[T] = V(T_m)$. A dominating set T of $V(T_m)$ is a secure dominating set of T_m if for each $u \in V(T_m)\setminus T$, there exists $v \in T$ such that $uv \in E(T_m)$ and the set $(T\setminus\{u\})\cup\{v\}$ is a dominating set of T_m. The minimum cardinality of a secure dominating set of T_m, denoted by $\gamma_s(T_m)$ is called a secure domination number of graph T_m. A secure dominating number $\gamma_s(T_m)$ of graph T_m is a triangular secure domination number if $\gamma_s(T_m)$ is a triangular number. In this paper, a combinatorial formula for triangular secure domination number of graph T_m was constructed. Furthermore, the said number was evaluated in relation to perfect numbers.

Keywords: triangular secure domination number; Mersenne prime; perfect number.

2010 Mathematics Subject Classification: 05C69.

Abstrak

Misal $T_m = (V(T_m), E(T_m))$ adalah graf triangular grid ingkat $m \in \mathbb{N}$. Order dari graf T_m disebut bilangan triangular. Suatu himpunan bagian T dari $V(T_m)$ adalah himpunan yang mendominasi T_m jika untuk semua $u \in V(T_m)\setminus T$ terdapat $v \in T$ sehingga $uv \in E(T_m)$, yaitu, $N[T] = V(T_m)$. Suatu himpunan T yang mendominasi $V(T_m)$ adalah himpunan mendominasi aman dari T_m jika untuk setiap $u \in V(T_m)\setminus T$ terdapat $v \in T$ sehingga $uv \in E(T_m)$ dan himpunan $(T\setminus\{u\})\cup\{v\}$ adalah himpunan yang mendominasi T_m. Kardinalitas minimum dari suatu himpunan mendominasi aman dari T_m, dinotasikan dengan $\gamma_s(T_m)$, bilangan dominasi aman dari graf T_m. Suatu bilangan dominasi aman $\gamma_s(T_m)$ dari graf T_m adalah bilangan dominasi aman tringular jika $\gamma_s(T_m)$ adalah bilangan triangular. Dalam paper ini dikonstruksi suatu formula kombinatorial untuk bilangan dominasi aman triangular dari suatu graf T_m. Lebih lanjut, bilangan tersebut dievaluasi hubungannya dengan bilangan sempurna.

Kata kunci:.bilangan dominasi aman triangular; prim Mersenne; bilangan sempurna

1. INTRODUCTION

Consider a *connected graph* $G = (V(G), E(G))$ and let $u \in V(G)$. The *neighborhood* of u is the set $N_G(u) = N(u) = \{v \in V(G): uv \in E(G)\}$. If $A \subseteq V(G)$, then the *open neighborhood* of A is the set $N_G(A) = N(A) = \bigcup_{u \in A} N_G(u)$. The *closed neighborhood* of A is denoted and defined by $N_G[A] = N[A] = A \cup N(A)$. For other concepts in graph theory, readers may refer to [1] [2] [3]. A subset T of $V(G)$ is a minimum dominating set in graph G if for every $u \in V(G)\backslash T$, there exists $v \in T$ such that $uv \in E(G)$, that is, $N[T] = V(G)$. The smallest cardinality of a dominating set T is called *dominating number* of graph G denoted by $\gamma(G)$. Domination concepts in graphs and some of its variants can be found in [4] [5] [6] [7] [8]. A vertex $x \in T$ is said to T-defend u, where $u \in V(G)\backslash T$, if $xu \in E(G)$ and $(T\backslash\{x\}) \cup \{u\}$ is a dominating set in G. A dominating set T is a *secure dominating set* if for every $u \in V(G)\backslash T$, there exists $v \in T$ such that v T-defends u. The rigorous concepts of secure domination number of graphs had been studied in [9] [10] [11] [12]. Let $T_m = (V(T_m), E(T_m))$ be a *triangular grid graph* where $m \in \mathbb{N}$. A graph T_m is a *sub graph* of a tiling of the plane with *equilateral triangles* defined by the finite number of triangles called cells. The order of graph T_m is a triangular number, that is, $|T_m| = \frac{m(m+1)}{2}$, where m is the m^{th} triangular number in graph T_m [13] [2] [14]. Figure 1 below shows the illustration of triangular grid graphs T_4 and T_5.

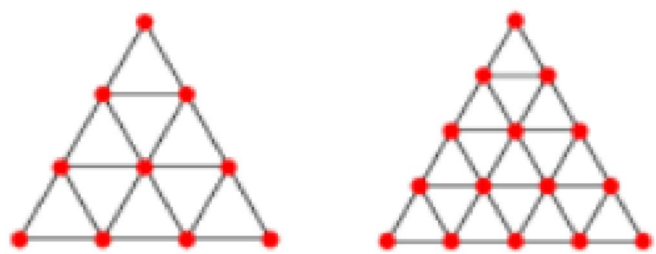

Figure 1. Triangular grid graphs T_4 and T_5, respectively.

The minimum cardinality of a secure dominating set of T_m, denoted by $\gamma_s(T_m)$ is called a secure domination number of T_m. A secure dominating number $|T| = \gamma_s(T_m)$ of graph T_m is a *triangular secure domination number* if $\gamma_s(T_m)$ is a triangular number. In number theory, *Mersenne primes* are type of prime numbers that can be derived using the formula $M_p = 2^p - 1$, where p is a prime number. Also, a *perfect number* is a positive integer of the form $P(p) = 2^{p-1}(2^p - 1)$, where p is prime and $2^p - 1$ is a Mersenne prime. Further, a perfect number can be written as the sum of its proper positive divisors, i.e., a number that is half the sum of all of its positive divisors. Let n be a positive integer and $\sigma(n)$ be the sum of all the positive divisors of n. So, we have $\sigma(n) = \sum_{d|n} d$, where d is a positive divisor of n. Hence, we say that n is a perfect number if $\sigma(n) = 2n$ by definition of perfect number [15] [16]. In this paper, we developed a combinatorial formula that determines the triangular secure domination number of triangular grid graph T_m. Furthermore, triangular secure domination number was investigated in relation to the intriguing numbers namely: Mersenne prime and perfect number.

2. RESULTS

The following Remark 2.1 is immediate from the definition of secure domination number of path P_n, where n is a positive integer.

Remark 2.1. [10] Let $G = P_n$ be a path of order $n \in \mathbb{N}$. Then,

$$\gamma_s(G) = \begin{cases} \frac{n+1}{2} & \text{if } n \equiv 1(mod2) \\ \frac{n}{2} & \text{if } n \equiv 0(mod2) \end{cases}$$

The next Theorem 2.2 is a direct consequence from Remark 2.1. The Theorem determines the secure domination number of graph T_m where m is an odd positive integer which denotes as the level of graph. The Theorem concludes that the secure domination number of graph T_m is the $\left(\frac{m+1}{2}\right)^{th}$ triangular number.

Theorem 2.2. Let T_m be a triangular grid graph of level $m \in \mathbb{N}$. If $m \equiv 1(mod\ 2)$, then $\gamma_s(T_m) = \frac{1}{8}(m^2 + 4m + 3)$ and $\gamma_s(T_m)$ is the $\left(\frac{m+1}{2}\right)^{th}$ triangular number.

Proof: Suppose that T_m is a triangular grid graph and $m \equiv 1(mod\ 2)$, then the Theorem will be proven with the following two cases:

Case 1: If $m = 1$, then $\gamma_s(T_m) = 1$ and obviously the hypothesis holds.

Case 2: Let $m \geq 3$. Now, consider the odd slanting paths of graph T_m, that is a sequence of paths $\{P_n\}$ where $n \in \{1, 3, \ldots, m\}$. By Remark 2.1, we obtained the series of secure domination number of odd slanting paths of graph T_m as follows:

$$\sum_{n=1}^{m} \gamma_s(P_n) = \sum_{i=1}^{\frac{m+1}{2}} \left\lceil \frac{n+1}{2} \right\rceil = \sum_{i=1}^{\frac{m+1}{2}} i = \frac{\left(\frac{m+1}{2}\right)\left(\frac{m+1}{2}+1\right)}{2} = \frac{1}{8}(m^2 + 4m + 3).$$

Since every vertex v in an even slanting paths is dominated by the one of the dominating vertex in an odd slanting paths, that is, $\forall\ v \in V(P_{n+1}) \subset V(T_m)\backslash T$, there exist $x \in V(P_n) \subset T$ such that $vx \in E(T_m)$, then T is a dominating set in graph T_m. Furthermore, any vertex $y \in T$ is a secure dominating vertex since for all $w \in V(T_m)\backslash T$, $wy \in E(T_m)$, and $(T\backslash\{y\}) \cup \{w\}$ is dominating set in graph T_m. Hence, it is concluded that $\gamma_s(T_m) = \frac{1}{8}(m^2 + 4m + 3)$ and it is the $\left(\frac{m+1}{2}\right)^{\text{th}}$ triangular number. This completes the proof. ■

Next result is a Corollary which is immediate from Theorem 2.2. This Corollary will determine the secure domination number of graph T_m where m is an even positive integer.

Corollary 2.3. Let T_m be a triangular grid graph of level $\mathrm{m} \in \mathbb{N}$. If $m \equiv 0(mod\ 2)$, then $\gamma_s(T_m) = \gamma_s(T_{m-1}) + \gamma(P_m)$.

Proof: It is clear that the first $(m-1)$ levels of graph T_m has $\gamma_s(T_{m-1})$ secure domination number by Theorem 2.2. This configuration already dominates the whole graph T_m, that is, for all $u \in$

$T, N[T] = V(T_m)$. But this dominating set is not a secure domination. So, it follows that to make it a secure dominating set, we will have an additional dominating vertex for m^{th} level of graph T_m, that is, $\gamma(P_m)$. Hence, $\gamma_s(T_m) = \gamma_s(T_{m-1}) + \gamma(P_m)$. This completes the proof. ∎

The following Remark 2.4 shows the domination number of path P_m of order $m \geq 1$, while Remark 2.5 shows the value of $\gamma_s(T_m)$ where $m \equiv 0(mod\ 2)$, which are immediate results from Corollary 2.3 and Remark 2.4.

Remark 2.4. [7] Let P_m be a path of order $\mathrm{m} \geq 1$. Then, $\gamma(P_m) = \left\lceil \frac{m}{3} \right\rceil$.

Remark 2.5. Let T_m be a triangular grid graph of level $m \in \mathbb{N}$. If $m \equiv 0(mod\ 2)$, then $\gamma_s(T_m) = \frac{1}{8}(m^2 + 2m) + \left\lceil \frac{m}{3} \right\rceil$.

The following Theorem and Corollary below are direct consequences of definition of perfect number and Theorem 2.2.

Theorem 2.6. Let $\frac{m+1}{2} = 2^t - 1$, where $m \equiv 1(mod\ 2)$ is the level of triangular grid graph T_m and $t \in \mathbb{N}$. Then, $\frac{m+1}{2}$ is a prime number for some prime number t if and only if $\gamma_s(T_m)$ is a perfect number.

Proof:

($\Longrightarrow$) Suppose that $\frac{m+1}{2} = 2^t - 1$, where t is a prime number and $m \equiv 1(mod\ 2)$. By Theorem 2.2, it directly follows that

$$\gamma_s(T_m) = \frac{1}{8}(m^2 + 4m + 3) = \frac{\left(\frac{m+1}{2}\right)\left(\frac{m+1}{2} + 1\right)}{2} = 2^{t-1}(2^t - 1).$$

Since $\frac{m+1}{2}$ is a prime number for some prime t, then, we have $\sigma\left(\frac{m+1}{2}\right) = \sigma(2^t - 1) = 2^t$. So it follows that $\sigma(\gamma_s(T_m)) = 2^t(2^t - 1) = 2\gamma_s(T_m)$ and follows directly that $\gamma_s(T_m)$ is a perfect number.

($\Longleftarrow$) For the converse, we let $\gamma_s(T_m)$ be a perfect number and $\frac{m+1}{2} = 2^t - 1$. By Theorem 2.2, we have $\gamma_s(T_m) = 2^{t-1}(2^t - 1)$. Since the greatest common divisor of 2^{t-1} and $(2^t - 1)$ is 1, then $\sigma(\gamma_s(T_m)) = \sigma(2^{t-1})\sigma(2^t - 1) = (2^t - 1)\sigma(2^t - 1)$. By the definition of perfect number, we have, $(2^t - 1)\sigma(2^t - 1) = 2^t(2^t - 1)$ and it simply follows that $\sigma(2^t - 1) = 2^t$. Thus, it clearly follows that $2^t - 1$ is a Mersenne prime number for some prime t and this completes the proof. ∎

Corollary 2.7. Let T_m be a triangular grid graph of level $m \in \mathbb{N}$. Then $\gamma_s(T_m) + \gamma_s(T_{m-2}) = [\gamma_s(T_m) - \gamma_s(T_{m-2})]^2$.

Let V_2, V_4 and V_6 be sets of vertices of triangular grid graph T_m having the degree of 2, 4 and 6, respectively, that is, $V(T_m) = V_2 \cup V_4 \cup V_6$ and $V_2 \cap V_4 \cap V_6 = \emptyset$. Then, the following Lemma is constructed.

Lemma 2.8. For any triangular grid graph T_m where $m \in \mathbb{N}$, the following statements hold:

i. $|V_2| = 3$ if $m \geq 2$;
ii. $|V_4| = 3m - 6$ if $m \geq 3$; and
iii. $|V_6| = \frac{1}{2}(m^2 - 5m + 6)$ if $m \geq 4$.

Proof: By counting the vertices of triangular grid graph T_m in relation to the sets V_2, V_4 and V_6, then, it is clear that Lemma 2.8 holds. ■

The next corollary which is immediate from Lemma 2.8, shows the relationships of sets V_2, V_4 and V_6 in relation to the interval values of positive integer m.

Corollary 2.9. Let T_m be a triangular grid graph with order $m \in \mathbb{N}$. Then the following holds:

i. $|V_6| < |V_2| = 3 \leq |V_4|$ if $m = 3 \; or \; 4$;
ii. $3 = |V_2| \leq |V_6| \leq |V_4|$ if $5 \leq m \leq 9$; and
iii. $3 = |V_2| < |V_4| < |V_6|$ if $m \geq 10$.

Next, the following result is quick from Remark 2.1, Theorem 2.2 and Lemma 2.8, shwoing the number of dominating vertices of graph T_m in the sets V_2, V_4 and V_6.

Theorem 2.10. Suppose that T_m be a triangular grid graph with order $m \equiv 1(mod\ 2)$ and $T \subset V(T_m)$ be the set of triangular secure dominating vertices. Then the following holds:

i. $|V_2 \cap T| = 3$ if $m \geq 3$;
ii. $|V_4 \cap T| = \frac{1}{2}(3m - 9)$ if $m \geq 5$; and
iii. $|V_6 \cap T| = \frac{1}{8}(m^2 - 8m + 15)$ if $m \geq 7$.

Proof: By Remark 2.1 and Theorem 2.2, it is clear that $V_2 \subseteq T$. So, it follows that $|V_2 \cap T| = 3$ by Lemma 2.8, thus (*i*) holds. Same argument can be applied to (*ii*) and (*iii*). And this completes the proof. ■

3. CONCLUSIONS

In this study, we developed a combinatorial formula to determine the triangular secure domination number of graph T_m, that is, $\gamma_s(T_m) = \frac{1}{8}(m^2 + 4m + 3)$ where m is odd. Result reveals that $\gamma_s(T_m)$ is the $\left(\frac{m+1}{2}\right)^{th}$ triangular number for $m \equiv 1(mod\ 2)$. Also, it is concluded that $\gamma_s(T_m)$ can be characterized with the concept of perfect numbers, that is, by considering $\frac{m+1}{2} = 2^t - 1$, where $m \equiv 1(mod\ 2)$ is the level of triangular grid graph T_m and $t \in \mathbb{N}$. So, we have $\frac{m+1}{2}$ is a Mersenne prime number for some prime number t if and only if $\gamma_s(T_m)$ is a perfect number. Futhermore, we have generated some important results regarding the triangular secure domination number of graph T_m where $m \equiv 1(mod\ 2)$.

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