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Aggregate Risk Model and Risk Measure-Based Risk Allocation

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Abstract

In actuarial modeling, aggregate risk is known as more attractive rather than individual risk. It has, however, usual difficulty in finding (the exact form of) joint probability distribution. This paper considers aggregate risk model and employ translated gamma approximation to handle such distribution function formulation. In addition, we deal with the problem of risk allocation in such model. We compute in particular risk allocation based on risk measure forecasts of Value-at-Risk (VaR) and its extensions: improved VaR and Tail VaR. Risk allocation shows the contribution of each individual risk to the aggregate. It has a constraint that the risk measure of aggregate risk is equal to the aggregate of risk measure of individual risk.

Keywords: allocation methods; tail-value-at-risk; translated gamma approximation.

Abstrak

Risiko agregat merupakan kajian yang lebih menarik dalam pemodelan aktuaria, dibandingkan dengan risiko individu. Namun fungsi distribusi risiko agregat sulit ditentukan bentuk eksaknya. Artikel ini membahas mengenai model risiko agregat dan menggunakan metode aproksimasi Translasi Gamma untuk menentukan fungsi distribusi risiko agregat. Berdasarkan fungsi distribusi tersebut, dapat diprediksi alokasi risiko agregat. Metode alokasi risiko agregat diterapkan pada ukuran risiko Value-at-Risk (VaR) dan pengembangannya: improved VaR dan Tail-VaR. Alokasi risiko menyatakan nilai kontribusi setiap risiko individu terhadap ukuran risiko agregat. Jumlahan atau agregat dari setiap alokasi risiko individu sama dengan ukuran risiko agregat.

Kata kunci: aproksimasi Translasi Gamma, alokasi risiko, *Tail-Value-at-Risk*.

1. INTRODUCTION

Risk modeling in finance and insurance industries has received much attention significantly in recent years from both academia and practitioners. Their interests lie from its attractive statistical derivation to the use of forecasting (for reserving, in particular) e.g. Syuhada [1]. In this paper, we are interested in modeling of collection of individual risk namely aggregate risk. Such aggregate model may be constructed through several possible scenarios: the number of random losses is either deterministic or stochastic; sequence of random losses may be (in)dependent and (not) identically distributed.

When applying aggregate risk model, we consider two potential problems. The first is usual difficulty of finding joint probability distribution in which we apply the method of translated gamma approximation [2]. The second is the problem of risk allocation based on risk measure forecast. Specifically, we employ Value-at-Risk (VaR) forecast and its extensions: improved VaR and Tail VaR. Risk allocation may be described as the contribution of each component for aggregate model with the constraint of: risk measure forecast of aggregate risk is equal to the aggregate of risk measure of individual risk.

The choice of risk measure of VaR greatly relies on the common use in practice. It reflects the maximum tolerated risk at certain confidence level and time period, e.g. Nieto and Ruiz [3], for latest review on VaR and its backtesting, and Chen [4]. VaR generally tell us a warning before the worse risk occurs as well as a preparation of capital. Although VaR is widely-used, it is not a coherent risk measure. To deal with this issue, we use an extension of VaR of what so-called Tail VaR or TVaR. It is the expected value of losses beyond VaR.

The remainder of this paper, after Introduction, is organized as follows. Section 2 describes aggregate risk model and approximation of translated gamma for joint distribution. Our main result is provided in Section 3 which includes methods of risk allocation. Concluding remark is in Section 4.

2. AGGREGATE RISK MODEL AND ITS PROBABILITY DISTRIBUTION

As stated above, risks may come more attractive and beneficial in the form of aggregate model rather than individual. Suppose that X_Θ represents aggregate risk of collection of non-negative random losses $\{X_i : i = 1,2,\cdots,n\}$ given by

$$X(\Theta) = \theta_1 X_1 + \cdots + \theta_n X_n = \sum_{i=1}^{n} \theta_i X_i,$$

where $\theta_i \in \mathrm{R}$ and each random loss may have correlation with other i.e. $\rho(X_i;X_j) \in (-1, 1)$, $i,j = 1,2,\cdots,n$ and $i \neq j$. Note that the coefficient θ is usually assumed $\theta_i = 1$, $\forall i$, see e.g. Dhaene et al. [5], and Kim and Kim [6].

The usual difficulty in aggregate model is in finding joint distribution, particularly when the individual risk component is not independent with other. In what follows, we describe a DV method [2] of translated gamma approximation for joint distribution function.

TRANSLAted GAMMA APPROXIMATION

In general, non-negative random losses are skewed to the right (skewness $\gamma > 0$) and are unimodal. Thus, they have roughly the shape of a gamma distribution. Apart from the usual parameters α and β, approximation of the distribution function of $X(\Theta)$ is carried out by allowing a shift over a distance x_0, e.g. Kaas et al. ([7] pp. 32). Then, choose α, β, and x_0 in order to make the approximated random losses has the same first three moments as of $X(\Theta)$.

The distribution function of the translated gamma approximation, according to Dhaene and Vyncke [2], is given by

$$F_{X(\Theta)}(x) \approx G(x - x_0; \alpha, \beta),$$

where

$$G(x;\alpha,\beta)=\frac{1}{\Gamma(\alpha)}\int_0^x y^{\alpha-1}\beta^{\alpha}e^{-\beta y}dy$$

and $x\geq 0$. Note that $G(x;\alpha,\beta)$ is the gamma distribution function. To apply the approximation, choose parameters α, β, and x_0 so that the first, second, and third central moments of $X(\Theta)$ equal the corresponding moments for the translated gamma distribution. Therefore, $\mu=x_0+\frac{\alpha}{\beta},\sigma^2=\frac{\alpha}{\beta^2},\gamma=2/\sqrt{\alpha}$. We obtain

$$\alpha=\frac{4}{\gamma^2},\beta=2/(\gamma\sigma),x_0=\mu-\frac{2\sigma}{\gamma},$$

where μ, σ^2, γ are mean, variance, and skewness of $X(\Theta)$, respectively.

Without loss of generality, we consider the aggregate risks of dependent identical exponential(λ) random losses with $n=2$ and $\theta_1=\theta_2=1$, then $X(\Theta)=X_1+X_2$. The joint probability function is obtained through the first derivative of joint distribution function based on FGM (Farlie Gumbel Morgenstern) family. Thus, we have

$$f_{X_1,X_2}(x_1,x_2)=f_{X_1}(x_1)f_{X_2}(x_2)\left[1+\phi\left(1-2F_{X_1}(x_1)\right)\left(1-2F_{X_2}(x_2)\right)\right]$$

which is equal to

$$\lambda^2\exp(-\lambda(x_1+x_2))[1+\phi(1-2(\exp(-\lambda x_1)+\exp(-\lambda x_2))+4\exp(-\lambda(x_1+x_2)))],$$

where ϕ is dependence parameter associated with Pearson's correlation ρ. For the case of exponential random losses, we have

$$\rho=\frac{E[X_1X_2]-E[X_1]E[X_2]}{\sqrt{Var(X_1)Var(X_2)}}=\frac{1/\lambda^2(1+\phi/4)-1/\lambda^2}{1/\lambda^2}$$

and thus we obtain $\phi=4(\rho-1)$. In addition, the expectation of $E[X_1^mX_2^n]$ for $m,n=1,2$ are, respectively $E(X_1^1X_2^1)=\rho/\lambda^2$ and $E(X_1^2X_2^1)=E(X_1^1X_2^2)=\rho/\lambda^3$ and these are easy to find. Thus, we obtain distribution function of the translated gamma approximations of $X(\Theta)$ for the case of two dependent identical exponential(λ) sum,

$$F_{X(\Theta)}(x)\approx G(x-x_0;\alpha,\beta)=\frac{1}{\Gamma(\alpha)}\int_0^{x-x_0}y^{\alpha-1}\beta^{\alpha}\exp(-\beta y)\,dy,$$

for $x-x_0\geq 0$, where

$$\alpha=\frac{32\rho^3}{(4-6\rho)^2},\ \beta=\frac{2\lambda\sqrt{8\rho^3}}{(4-6\rho)\sqrt{2\rho}},\ x_0=\frac{2(4-6\rho)-8\rho^2}{\lambda(4-6\rho)}.$$

We illustrate in the following Figure 1 of distribution function and its translated gamma approximation. It is shown several examples of known distributions.

3. RISK MEASURE-BASED RISK ALLOCATION

VALUE-AT-Risk AND ITS EXTENSION

For either a single random risk or aggregate risk, a risk measure of Value-at-Risk (VaR) provides a value of such risk with the condition that its distribution function is greater than or equal certain confidence level. VaR describes the maximum possible loss over a given time interval, with a given confidence level e.g. Chen [4]. One may say that VaR is a critical value of loss distribution before it reaches the worst risk. VaR may also be considered as a system for finance and insurance industries.

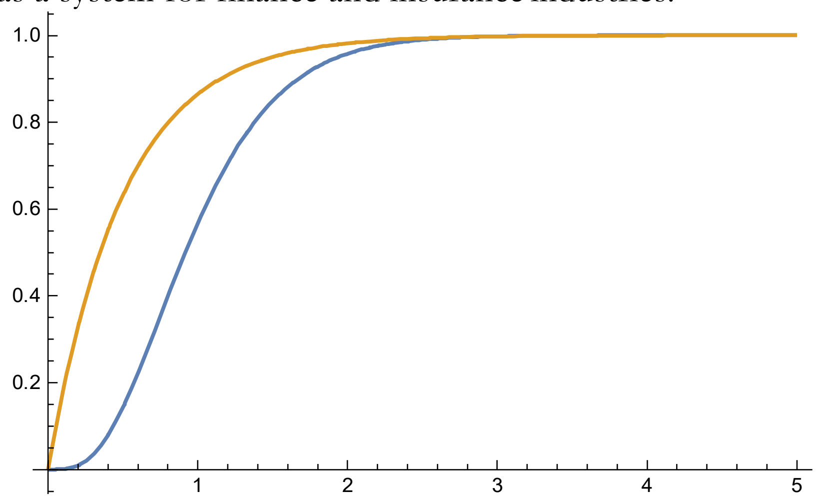

Figure 1. The distribution function and its translated gamma approximation.

In practice, finding VaR forecast may face difficulty in seeking appropriate loss distribution. In particular, we may not obtain the closed representative of inverse of distribution function. To handle this problem, VaR is formulated as

$$\vartheta(X;\delta) = inf\{x: F_X(x) = P(X \leq x) \geq \delta\}$$

where δ denotes confidence level. The accuracy of VaR forecast, via its coverage probability, is typically bounded to $O(n^{-1})$.

Kabaila and Syuhada [8] [9] provided a method to adjust VaR forecast with better coverage properties. The resulted VaR forecast, namely improved VaR, is basically obtained by absorbing the $O(n^{-1})$. term such that its coverage probability is bounded to $O\left(n^{-3/2}\right)$. Syuhada [1] shows the calculation of improved VaR forecast for the case of heteroscedastic processes.

Although VaR is widely-used, it is not a coherent risk measure (Artzner et al., [10]). Therefore, alternative coherent risk measure of Tail VaR is presented. Note that Tail VaR may be named is Conditional VaR or Expected Shortfall in literature. Basically, TVaR forecast calculates expected value or mean for losses beyond VaR i.e.

$$\zeta(X;\delta) = E\big(X|X > \vartheta(X;\delta)\big) = \frac{1}{1-\delta}\int_{\delta}^{1} \vartheta(X;\varepsilon)d\varepsilon.$$

TVaR contains more information than VaR by accounting entire right-sided tail starting from VaR of the loss distribution. We will apply both VaR and TVaR forecasts for aggregate risk model by noting that the distribution of aggregate claims is often skewed, e.g. Kaas et al. ([7] pp. 32) and the translated gamma approximation is employed.

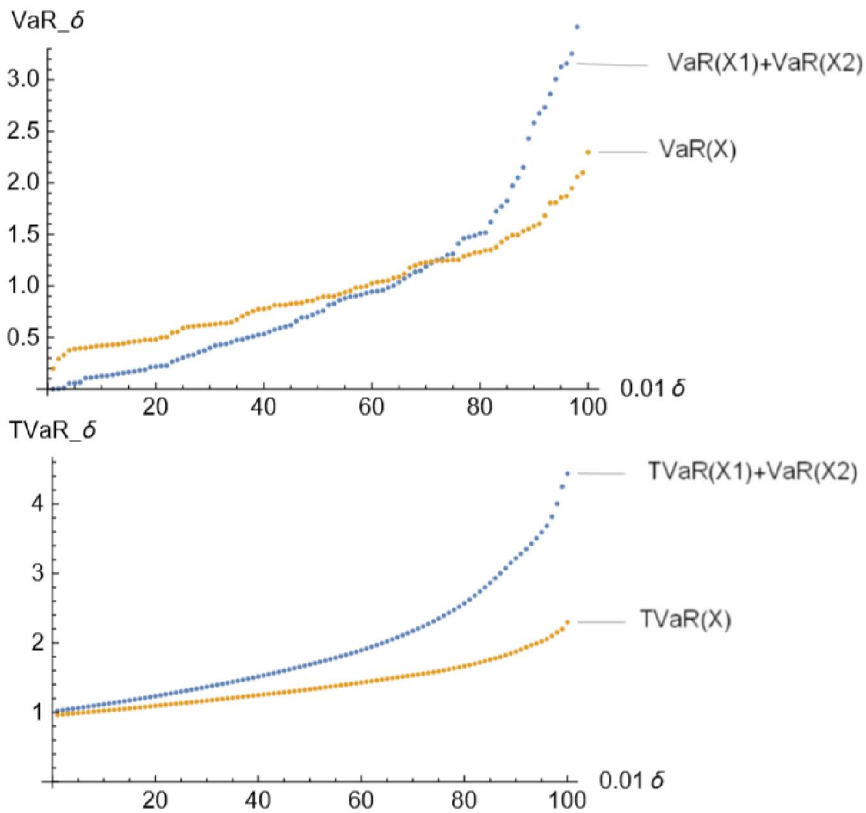

Figure 2. VaR dan TVaR forecasts for aggregate risk - the dependent exponential distribution case.

Figure 2 above shows VaR and TVaR calculations when we are considering of aggregate risk. It illustrates that VaR does not satisfy subadditivity property whilst TVaR does.

Risk allocation for aggregate risk

Risk allocation is a method performed to divide the aggregate risk down to individual risk in order to determine risk contribution. Specifically, we apply proportional and Euler's risk allocation, e.g. Tasche [11], Tasche [12], Kyselova [13]. We present these procedures under risk measure forecasts of VaR, improved VaR and TVaR. It is important to note that the problem of risk allocation is interesting and non-trivial since the simple sum of risk measure of individual losses is usually larger than the aggregate losses. Van Gulick et al. [14] defined two key properties for a feasible allocation method. The first is risk contribution should not exceed the stand alone risk. The second property is that it should not fall below the minimum loss that can occur from this position.

We assume that the random losses are $\{X_i: i = 1,2,\cdots,n\}$ and the aggregate risks is total risk is $X(\Theta) = \sum_{i=1}^{n} X_i$. When a risk measure is applied to aggregate risk model, we will decompose this measures to its units. In other words, we find the non-negative real numbers $\varrho_i(X(\Theta)), i = 1,\cdots,n$, such that

$$\varrho(X(\Theta)) = \sum_{i=1}^{n} \varrho_i(X(\Theta)).$$

There several risk allocation approaches. The first is proportional allocation approach. The particular allocated risks are obtained by first choosing a risk measure and then attributing the proportion $\varrho_i(X(\Theta)) = K\varrho(X(\Theta))$ to each unit ; $i = 1,\cdots,n$. If the risk measure is law-invariant, the proportional allocation is not influenced by dependence between the risks X_i [13]. The standard and haircut allocation principles are based on allocating the aggregate risks, respectively, using standard deviation and VaR as proportion. It leads to the formula of the allocation

$$\varrho_i(X(\Theta)) = \frac{sd(X_i)}{\sum_{j=1}^{n} sd(X_i)} \varrho(X(\Theta))$$

and

$$\varrho_i(X(\Theta)) = \frac{\vartheta(X_i;\delta)}{\sum_{j=1}^{n} \vartheta(X_i;\delta)} \varrho(X(\Theta))$$

for $i = 1,2,\cdots,n$. It is clear that this principle does not make allowance for a dependence structure between the losses X_i of the individual business units. Furthermore, because VaR is not a subadditive risk measure, it may occur that the allocated amount exceeds the respective stand-alone measures $\vartheta(X_i;\delta)$.

The Euler allocation principle also known as the gradient allocation principle. Euler allocation principle is the only per-unit capital allocation principle suitable for performance measurement [11]. For the Euler allocation, which is the predominated method for risk allocation in aggregate risks, the two key properties are fulfilled. If a risk measure is continuously differentiable and positive homogeneous, the Euler allocations are given by

$$\varrho^E(X_i|X(\Theta)) = \frac{d\varrho^E(X_i|X(\Theta))}{dh}(X(\Theta) + hx_i)\Big|_{h=0}$$

$\forall i = 1,\cdots,n$ with $\varrho(X(\Theta)) = \sum_{i=1}^{n} \varrho^E(X_i|X(\Theta))$.

According to Tasche [15], we have allocation formulas for the derivation of the Euler risk contributions under conditions which the corresponding quantiles are differentiable. This is important in order to determine the contribution of quantile-based risk measures as VaR and TVaR in the form of the partial derivative.

Table 1. Risk allocation methods and notations

Allocation Method	RM	$\varrho(X_i)$	$\varrho_i\big(X(\Theta)\big)$	Reference
Standard	$\varrho(X_i)$	$\varrho(X_i)$	$\frac{sd(X_i)}{\sum_{j=1}^{n} sd(X_j)}\varrho\big(X(\Theta)\big)$	Kyselova [13]
Haircut	$\varrho(X_i)$	$\varrho(X_i)$	$\frac{\vartheta(X_i;\delta)}{\sum_{j=1}^{n}\vartheta(X_j;\delta)}\varrho\big(X(\Theta)\big)$	Kyselova [13]
Euler	VaR	$\vartheta(X_i;\delta)$	$E[X_i \mid X(\Theta)=\vartheta(X(\Theta);\delta)]$	Tasche [11]
Euler	$TVaR$	$\zeta(X_i;\delta)$	$E[X_i \mid X(\Theta)>\vartheta(X(\Theta);\delta)]$	Tasche [15]

The risk allocation for the above-mentioned risk measures are listed in Table 1. Formula for the VaR using Euler allocation is obtained by computing mean of losses X_i given that the overall loss $X(\Theta)$ equal to VaR. While for the TVaR using Euler allocation is obtained by computing mean of losses X_i given that the overall loss $X(\Theta)$ beyond VaR.

Table 2. Risk allocation based on risk measure forecasts.

	Standard	Haircut	Euler
$VaR^i_{0.95}(X_1)$	0.8470	0.8274	0.8937
$VaR^i_{0.95}(X_2)$	1.0119	1.0135	0.9652
Improved $VaR^i_{0.95}(X_1)$	1.2753	1.2744	1.2078
Improved $VaR^i_{0.95}(X_2)$	1.2945	1.2954	1.3620
$TVaR^i_{0.95}(X_1)$	0.7589	0.7364	0.9916
$TVaR^i_{0.95}(X_2)$	1.1206	1.1225	1.0309

We compute the risk allocations under translated gamma approximation suggested in Section 2. Assume that the random losses are dependent identical exponential (λ = 2), with ρ = 0.5. Risk allocation results are presented in Table 2. Allocations obtained using the standard allocation method and the haircut allocation are similar, but quite far from the allocation with the Euler distribution method. This might be due to the two methods are not considering dependence between individual risks whilst the Euler allocation method does.

To investigate the risk measure and its allocation obtained using the Euler method, we use Euler formulas in Table 1 over various quantile levels. The results are showed in Figure 3. Finally, we compare three alternative risk measures for X in Figure 4. As expected, all the risk measures increase over the quantile levels. For $0 < \delta < 0.75$, TVaR is the largest followed by the improved VaR and VaR, while improved VaR exceeds TVaR and VaR for $0.75 < \delta < 1$.

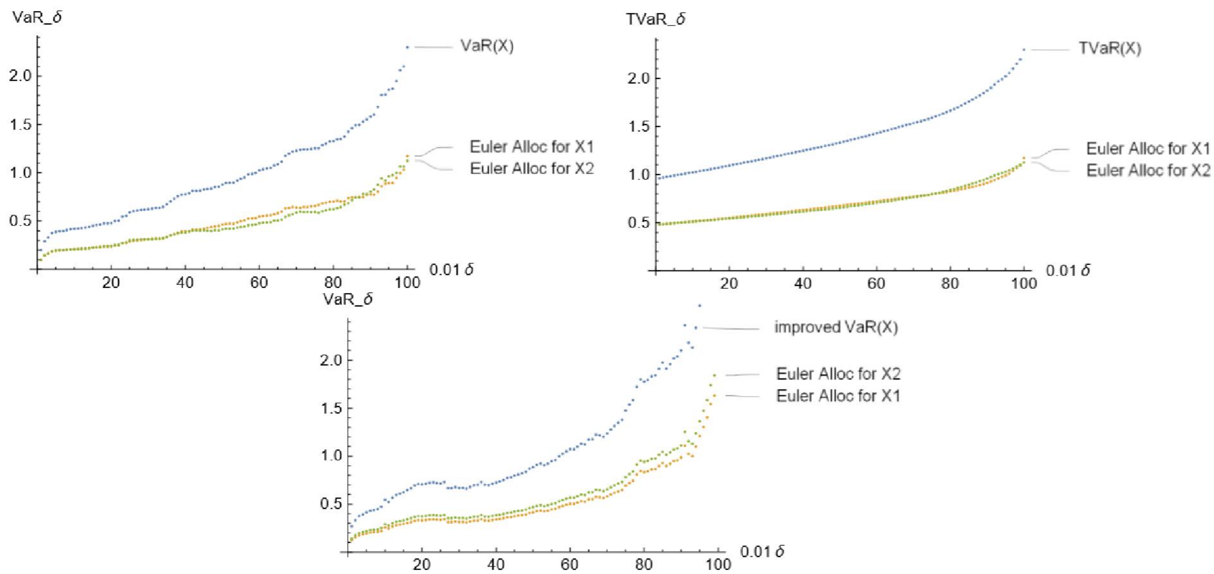

Figure 3. Euler risk allocation based on risk measure forecasts.

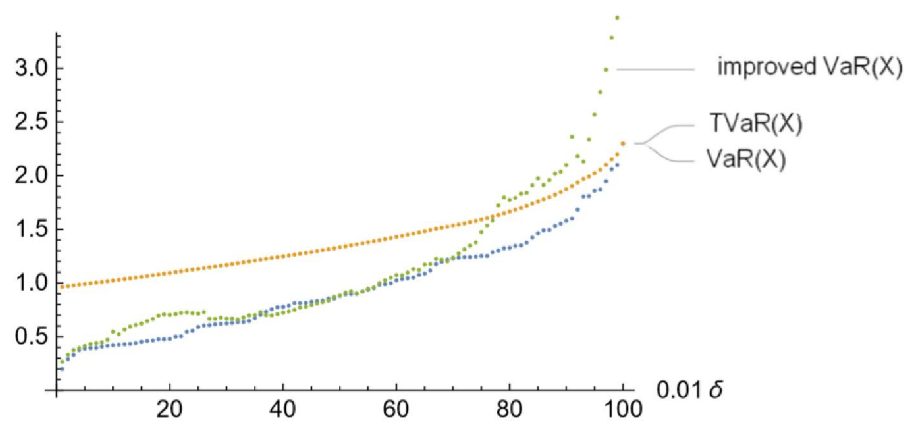

Figure 4. Risk measure forecasts-based for risk allocation.

Example of real application

We apply and compute risk allocation under translated gamma approximation suggested in Section 2 to real data. The data are Indosat (ISAT.JK) and Telkomsel (TLKM.JK) stock prices on 25 March 2019 to 25 March 2020. Each data are fitted into exponential distribution with parameters $\widehat{\lambda_1} = 0.000257$ and $\widehat{\lambda_2} = 0.00026$, with $\widehat{\rho} = 0.3$. Note that both parameters have almost the same value so that we assume $\widehat{\lambda} = 0.00026$.

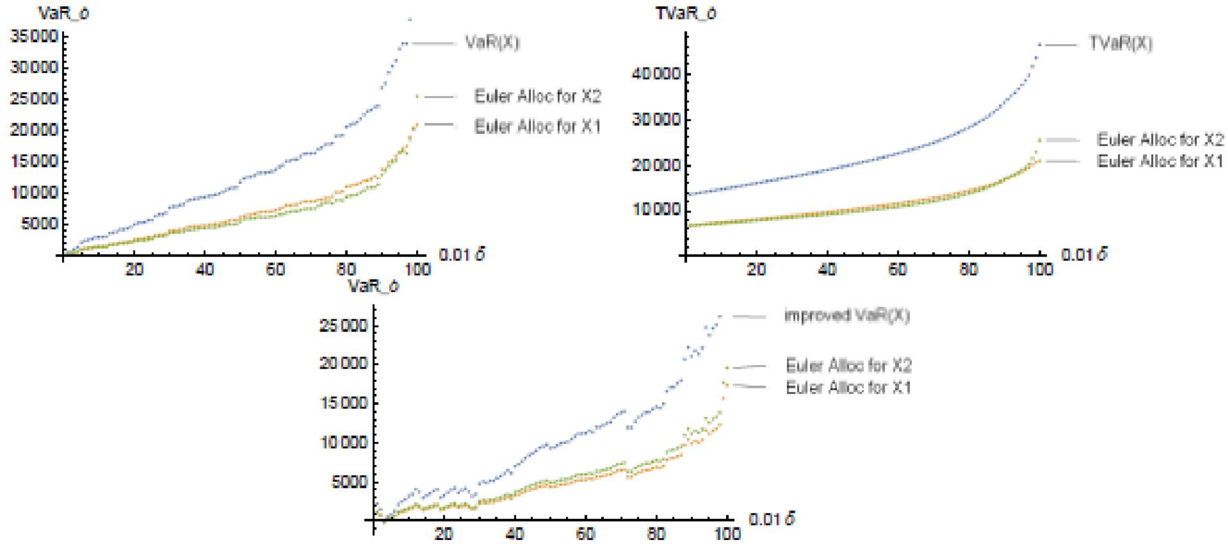

Figure 5. Euler risk allocation based on risk measure forecasts - real data case.

Table 3. Risk allocation based on risk measure forecasts for real data.

	Standard	Haircut	Euler
$VaR_{0.95}^{i}(X_1)$	14094.7	14056.1	14153.8
$VaR_{0.95}^{i}(X_2)$	15548.5	15587.1	15489.4
Improved $VaR_{0.95}^{i}(X_1)$	11060.9	11186.8	11274.3
Improved $VaR_{0.95}^{i}(X_2)$	12582.3	12456.4	12395.9
$TVaR_{0.95}^{i}(X_1)$	17014.4	17095.9	17244.6
$TVaR_{0.95}^{i}(X_2)$	18828.2	18746.7	18598.1

There is a considerable difference in value of prediction of VaR, improved VaR and TVaR under translated gamma approximation method. This might be due to the parameter $\hat{\lambda}$ which has little value. As a result, the risk allocation for each risk measure differs quite significantly. We may observe these in Table 3 for all three methods.

Furthermore, there are large differences between allocation obtained using the Euler method. The results are shown in Figure 5; since VaR forecast reaches 30000 for $\delta = 0.95$, the Euler allocation is around 15000. This also applied to TVaR and improved VaR, whose values reach 23,000 and 35,000 respectively. We compare three alternative risk measures for X from real data in Figure 6. As expected, there are large differences between all the risk measures with TVaR is the largest followed by VaR and the improved VaR.

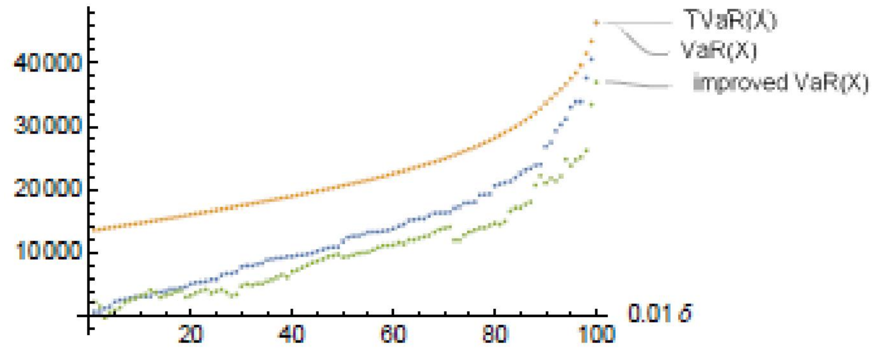

Figure 6. Risk measure forecasts-based for risk allocation - real data case.

4. CONCLUDING REMARK

The risk allocation for aggregate risks may be applied to heteroscedastic processes since financial data are most likely modeled with dynamic volatility. For this case, the use of Copula may be more intensive. It is also interesting to consider certain financial institution of business entity such as Cooperative in calculating risk measure forecast for each member.

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