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Some Notes on Relative Commutators

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Abstract

Let G be a group and $\alpha \in Aut(G)$. An α-commutator of elements $x, y \in G$ is defined as $[x, y]_\alpha = x^{-1}y^{-1}xy^\alpha$. In 2015, Barzegar et al. introduced an α-commutator of elements of G and defined a new generalization of nilpotent groups by using the definition of α-commutators which is called an α-nilpotent group. They also introduced an α-commutator subgroup of G, denoted by $D_\alpha(G)$ which is a subgroup generated by all α-commutators. In 2016, an α-perfect group, a group that is equal to its α-commutator subgroup, was introduced by authors of this paper and the properties of such group was investigated. They proved some results on α-perfect abelian groups and showed that a cyclic group G of even order is not *a*-perfect for any $\alpha \in Aut(G)$. In this paper, we may continue our investigation on α-perfect groups and in addition to studying the relative perfectness of some classes of finite p-groups, we provide an example of a non-abelian α-perfect 2-group.
Keywords: Auto-commutator subgroup; finite p-group; normal subgroup; perfect group.

Abstrak

Misalkan G grup dan $\alpha \in Aut(G)$. Suatu α-komutator dari unsur-unsur $x, y \in G$ didefinisikan sebagai $[x, y]_\alpha = x^{-1}y^{-1}xy^\alpha$. Pada tahun 2015, Barzegar et al. memperkenalkan α-komutator dari unsur-unsur di G dan mendefinisikan sebuah perumuman baru dari grup-grup nilpoten dengan menggunakan definisi dari α-komutator yang dinamakan grup α-nilpoten. Mereka juga memperkenalkan suatu subgrup α-komutator dari G yang dilambangkan dengan $D_\alpha(G)$ yang merupakan subgrup yang dibangun dari semua α-komutator. Pada tahun 2016, grup α-sempurna, yaitu grup yang subgrup α-komutatornya sama dengan grup itu sendiri, diperkenalkan oleh penulis paper ini dan sifat-sifat grup tersebut juga diselidiki. Mereka membuktikan beberapa sifat dari grup abel α-sempurna dan memperlihatkan bahwa suatu grup siklis G dengan order genap bukan grup *a*-sempurna untuk setiap $\alpha \in Aut(G)$. Di paper ini kita akan melanjutkan investigasi kita pada grup-grup α-sempurna dan sebagai tambahan dalam mempelajari kesempurnaan relatif dari kelas-kelas dari p-grup berhingga, kita akan melihat contoh dari 2-grup α-sempurna yang non abel.
Kata kunci: subgrup auto-komutator; p-grup berhingga; subgrup normal; grup sempurna.

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1. INTRODUCTION

In 1994, an auto-commutator $[x,\alpha]=x^{-1}x^{\alpha}$ of elements $x\in G$ and $\alpha\in Aut(G)$ was introduced by Hegarty, [1]. If α_g is an inner automorphism such that $x^{\alpha g}=g^{-1}xg$ then auto-commutator $[x,\alpha_g]=x^{-1}g^{-1}xg$ is the ordinary commutator of two elements $x,g\in G$. Hegarty generalized the definition of the center of G, $Z(G)=\{x\in G: x^{\alpha y}=x, \forall y\in G\}$ to the absolute center $L(G)=\{x\in G : x^{\alpha}=x, \forall\alpha\in Aut(G)\}$ of G. One can check that $L(G)$ is an characteristic subgroup of G which is contained in $Z(G)$. He also introduced the auto-commutator subgroup of G, denoted by $K(G)$, which is a characteristic subgroup generated by all auto-commutators. Clearly, the commutator subgroup G' is contained in $K(G)$. Investigation of the relative commutators are interesting for some authors, for instance Barzegar et al. [2] also introduced a new generalization of commutators with respect to a fixed automorphism of group G. Let $\alpha\in Aut(G)$, then an α-commutator of two elements $x,g\in G$ is defined as $[x,y]_{\alpha}=x^{-1}y^{-1}xy^{\alpha}$ which is equal to the ordinary commutator $[x,y]=x^{-1}y^{-1}xy$ whenever α is the identity automorphism. In [2], the subgroup which is generated by all α-commutators was denoted by $D_{\alpha}(G)$ and called α-commutator subgroup of G. It is not difficult to prove that $D_{\alpha}(G)$ is a normal subgroup of G that is contained in $K(G)$. Authors of [2] also introduced a new generalization of a nilpotent group G, which is called an α-nilpotent group for a fixed automorphism α of G. Here, we may present the definition of an α-nilpotent group G. We start by the definition of a lower central α-series. Put $\Gamma_1^{\alpha}(G)=G$ and $\Gamma_2^{\alpha}(G)=D_{\alpha}(G)$ and define inductively $\Gamma_{n+1}^{\alpha}(G)=[G,\Gamma_n^{\alpha}(G)]_{\alpha}=\langle[x,y]_{\alpha}: x\in G, y\in\Gamma_n^{\alpha}(G)\rangle, n\geq 1$. We can see that $\Gamma_n^{\alpha}(G)$ is a normal subgroup of G which is invariant under α and $\Gamma_{n+1}^{\alpha}(G)\leq\Gamma_n^{\alpha}(G)$, for all $n\geq 1$. Following normal series is called a lower central α-series $G\geq\Gamma_2^{\alpha}(G)\geq\cdots\geq\Gamma_n^{\alpha}(G)\geq\cdots$.

A group G is called an α-nilpotent group of nilpotency class n if $\Gamma_n^{\alpha}(G)=\{1\}$ and $\Gamma_{n+1}^{\alpha}(G)\neq\{1\}$. Clearly, if α is considered as the identity automorphism, then an α-nilpotent group is the ordinary one. In [2], it was proved that an α-nilpotent group is nilpotent, but the converse is not valid in general. For instance, authors proved that the cyclic group of order $n=p_1p_2...p_t$ is α-nilpotent if and only if α is the identity automorphism, for distinct primes $p_1,p_2,...,p_t$. Authors of [3] continued investigation on α-nilpotent groups and proved some new results on this new concept. For example, they proved that an extra special p-group, p is an odd prime number, is nilpotent with respect to a non-identity automorphism α but is not nilpotent relative to all its automorphisms. For an inner automorphism $\alpha_g\in Inn(G)$, we can see that nilpotency and α_g-nilpotency are equivalent. Therefore, we may ask the following question.

Question. Is there a non-inner automorphism α of nilpotent group G such that G is α-nilpotent?

This question was answered for finitely generated abelian groups, for more details see [3]. Actually, authors classified all finitely generated abelian groups which are nilpotent with respect to a non-inner automorphism. Furthermore, they proved some results on relative normal and absolute normal subgroups of some classes of finite groups. In [4], they introduced an α-perfect group G, a group which is equal to its α-commutator subgroup, for a fixed automorphism α of G. If G' is the ordinary commutator subgroup of G, then $G' \leq D_\alpha(G)$ for all $\alpha \in Aut(G)$. It follows that if G is a perfect group, then it is perfect with respect to all its automorphisms. One can check that an α-nilpotent group cannot be α-perfect, but the symmetric group of order $n!$, S_n is an example of a non-nilpotent group where is not α-perfect, because $D_\alpha(S_n) = (S_n)' = A_n$ for all $\alpha \in Aut(G)$. The relative perfectness of abelian groups was studied by authors of [4]. In this paper, we may continue our investigation on relative perfect groups and prove some new results on some classes of finite non-abelian p-groups.

2. RELATIVE PERFECT GROUPS

In this section, we recall the definition of an α-perfect group for a fixed automorphism α. At first, we present some results on relative perfect groups that were proved in [4]. Finally, we may add some new results on non-abelian relative perfect groups.

Definition 2.1. Let G be a group and $\alpha \in Aut(G)$. A group G is called an α-perfect group, whenever $G = D_\alpha(G)$.

Definition 2.2. If G is a finite group and $\alpha \in Aut(G)$, then a subgroup H of G is called an α-normal subgroup of G, denoted by $H \overset{\alpha}{\trianglelefteq} G$, if $g^{-1}hg^\alpha \in H$ for all $g \in G$ and $h \in H$. If H is α-normal with respect to all automorphisms $\alpha \in Aut(G)$, then G is called an absolute normal subgroup of G.

Lemma 2.3. ([4]) Let H be a subgroup of finite group G, then (i) if there exists an $\alpha \in Aut(G)$ such that $H \overset{\alpha}{\trianglelefteq} G$, then H is a normal subgroup of G, (ii) H is an absolute normal subgroup of G if and only if $K(G) \leq H$.

It might be important to find all proper absolute normal subgroups of given finite group G. In [3] and [4], the structure of absolute normal subgroups of some classes of finite groups were given. For instance, we have the following results.

Lemma 2.4. ([4]) If $G \cong Z_{2^n m}$ such than $(2, m) = 1$, then the proper subgroup H of G is absolute normal if and only if $H = 2G$.

Theorem 2.5. ([3]) (i) If $D_{2n} = \langle x, y : x^n = y^2 = 1, yxy = x^{-1} \rangle$, then $\langle x \rangle$ is the only proper absolute normal subgroup of D_{2n}. (ii) Semi-dihedral 2-group $SD_{2^{n+1}} = \langle x, y : x^{2n} = y^2 = 1, yxy = x^{-1} = x^{2^{n-1}-1} \rangle, n \geq 3$ has the only proper absolute normal subgroups given by $\langle x \rangle, \langle x^2 \rangle, \langle x^2, y \rangle, \langle x^2, yx \rangle$. (iii) Generalized quaternion 2-group $Q_{2^{n+1}}=\langle x,y : x^{2n}=1,\ x^{2n-1}=y^2,\ yxy^{-1}=x^{-1}\rangle, n\geq 3$ has the only proper absolute normal subgroup $\langle x\rangle$. (iv) Twisted dihedral 2-group $S_*D_{2^{n+1}}=\langle x,y : x^{2n}=y^2=1,\ yxy^{-1}=x^{2^{n-1}+1}\rangle, n\geq 3$ has the only proper absolute normal subgroup $\langle x^2, y\rangle$.

Theorem 2.6. ([3]) If p is an odd prime number and $M_n(p)=\langle x,y : x^{p^{n-1}}=y^p=1, xy=yx^{p^{n-2}+1}\rangle$, $n\geq 3$, then $M_n(p)$ does not possess a proper absolute normal subgroup.

Next lemma, talks about the existence of an α-normal subgroup in abelian α-perfect group G.

Lemma 2.7. ([4]) Let G be a finite abelian group. Then G is α-perfect if and only if G does not possess a proper α-normal subgroup.

If G is a finite cyclic group of order n, then $\alpha\in Aut(G)$ if and only if $x^{\alpha}=ux$ such that $(u,n)=1$, for all $x\in G$. We denote such α by α_u.

Lemma 2.8. ([4]) Let G be a cyclic group of order n and $\alpha_u\in Aut(G)$ be a non-identity automorphism. Then G is α_u-perfect if only if $(u-1,n)=1$.

By Lemma 2.8, we can conclude that there is no α-perfect cyclic group of even order, for all automorphisms α of such group. If p is an odd prime number, then $\mathrm{Z}_{p^r}, r>1$, is α_u-perfect for each $1<u<p$, but it is not α_{p+1}-perfect.

Now, we are ready to prove some new results on relative perfect groups.

Lemma2.9. If α_g is an inner automorphism and $\beta=\alpha\circ\alpha_g$, then G is α-perfect if and only if is β-perfect.

Proof. We can see that $[x,y]_{\beta}=[x,y]_{\alpha}[y,g]^{\alpha}$ and since $[y,g]\in G'\leq D_{\alpha}(G)$ and $D_{\alpha}(G)$ is α-invariant, then $[x,y]_{\beta}\in D_{\alpha}(G)$ and so $D_{\beta}(G)\leq D_{\alpha}(G)$. We can write $\alpha=\beta\circ\alpha_g^{-1}$ and prove $D_{\alpha}(G)\leq D_{\beta}(G)$. Now, we are done. ■

Example 2.10. (i) If G is isomorphic to one of the groups where are defined in Theorem 2.5, then G possesses a proper absolute normal subgroup. Furthermore, we know that $D_{\alpha}(G)\leq K(G)$, for all $\alpha\in Aut(G)$. So by Lemma 2.3, $D_{\alpha}(G)$ is a proper subgroup of G and G is not α-perfect for any $\alpha\in Aut(G)$. (ii) Assume that that $Q_8=\langle xy : x^4=1,\ x^2=y^2, yxy^{-1}=x^{-1}\rangle, n\geq 3$ and $\alpha\in Aut(Q_8)$ is an automorphism by argument $x^{\alpha}=y$ and $y^{\alpha}=xy$. Then $D_{\alpha}(Q_8)=Q_8$ and Q_8 is an α-perfect group.

By Theorem 2.6, if $G\cong M_n(p)$, then G does not possess any proper absolute normal subgroup. Here, we may prove that $M_n(p)$ is not α-perfect for any $\alpha\in Aut(G)$.

Theorem 2.11. If $G\cong M_n(p)=\left\langle x,y: x^{p^{n-1}}=y^p=1, xy=yx^{p^{n-2}+1}\right\rangle$ for an odd prime number p and $n\geq 3$, then G is not α-perfect for any $\alpha\in Aut(G)$.

Proof. We can see that $|G|=p^n, Z(G)=\langle x^p\rangle, G'=\left\langle x^{p^{n-2}}\right\rangle$. The automorphism group of G, $Aut(G)$ is equal to $\left\{\alpha_{ijk}: x^{\alpha_{ijk}}=x^iy^j, y^{\alpha_{ijk}}=x^{kp^{n-2}}y, 0\leq i\leq p^{n-1}-1, i\overset{p}{\neq}0, 0\leq j,k\leq p-1\right\}$. It is not difficult to see that $[x,x]_{\alpha_{ijk}}=x^{i-1}y^j, [x,y]_{\alpha_{ijk}}=x^{(k+1)p^{n-2}}, [y,x]_{\alpha_{ijk}}=x^{p^{n-2}}x^{i-1}y^j$, $[y,y]_{\alpha_{ijk}}=x^{kp^{n-2}}$. Therefore, $D_{\alpha_{ijk}}(G)=\left\langle x^{i-1}y^j, x^{p^{n-2}}\right\rangle$. If $i\overset{p}{\neq}1$, then $|x^{i-1}y^j|=|x|$, also $(i-1)p^{n-2}\overset{p^{n-1}}{\equiv}p^{n-2}$ and for a positive integer m we have $\left(x^{i-1}y^j\right)^m=x^{m(i-1)-(i-1)j\left(\frac{m(m-1)}{2}\right)p^{n-2}}y^{mj}$.

Now, if we put $m=p^{n-2}$, then since $n\geq 3$, $p^{n-2}\geq p$ and $(x^{i-1}y^j)p^{n-2}=x^{(i-1)p^{n-2}}=x^{p^{n-2}}$. It means that $x^{p^{n-2}}\in\left\langle x^{i-1},y^j\right\rangle$ and $D_{\alpha ij}(G)\in\left\langle x^{i-1},y^j\right\rangle$ is a proper subgroup of G, because $|x^{i-1}y^j|=p^{n-1}$. Now if, $i=1$ and $j\neq 0$, then $D_{\alpha ij}(G)\in\left\langle y^j, x^{p^{n-2}}\right\rangle$ and since $|y^j|=|x^{p^{n-1}}|=p$, then $D_{\alpha ij}(G)\neq G$. In case $j=0$, we have $D_{\alpha ij}(G)=G'\neq G$. If $i\neq 1$ and $i\overset{p}{\equiv}1$ then $i=1+lp^s$, where $(l,p)=1, s=1,2,\ldots,n-2$. In this case, G is α_{ijk}-nilpotent by Theorem 4.10 of [2] and so $D_{\alpha_{ijk}}(G)\neq G$. Hence, we are finished. ■

Theorem 2.12. Let p be an odd prime number. If $G=\left\langle a,b: a^{p^2}=b^{p^3}=1, a^{-1}ba=b^{p+1}\right\rangle$ is a p-group of order p^5 and nilpotency class three, then G is not α-perfect for any $\alpha\in Aut(G)$.

Proof. The automorphism group of G is $Aut(G)=\{\alpha_{z,\omega,\mu}: a^{\alpha_{z,\omega,\mu}}=a(a^zb^\omega)^\mu, b^{\alpha_{z,\omega,\mu}}=a^zb^\omega; zp\overset{p^2}{\equiv}0, \omega\overset{p}{\neq}0, \mu p^2\overset{p^3}{\equiv}0\}$. Let $\alpha=\alpha_{z,\omega,\mu}\in Aut(G)$, be an arbitrary automorphism such that $z=pt$ and $\mu=pk$ for some integers $t,k\in\mathbb{Z}$. Then (i) $[a,a]_\alpha=(a^{pt}b^\omega)^{pk}=a^{p^2tk}b^{\omega\frac{(p+1)^{p^2tk}-1}{(p+1)^{ptk}-1}}=b^{\omega\frac{(p+1)^{p^2tk}-1}{(p+1)^{ptk}-1}}$, (ii) $[a,b]_\alpha=a^{-1}b^{-1}aa^{pt}b^\omega=a^{-1}ab^{-p-1}a^{pt}b^\omega=b^{pt+\omega-p-1}$, (iii) $[b,a]_\alpha=b^{-1}a^{-1}ba(a^{pt}b^\omega)^{pk}=b^pb^{\omega\frac{(p+1)^{p^2tk}-1}{(p+1)^{ptk}-1}}$, and (iv) $[b,b]_\alpha=b^{-1}a^{pt}b^\omega=a^{pt}b^{-(p+1)^{pt}}=a^{pt}b^{-(p^2t+1)}$.

Assume that $\alpha \in D_\alpha(G)$, then there exists an integer j such that $a=(a^{pt}b^{-(p^2t+1)})^j$. Put $m=\frac{(p+1)^{ptj}-1}{(p+1)^{pt}-1}$, then since $Z(G)=\langle b^{p^2}\rangle$ we have $a=(a^{pt}b^{-1})^j b^{-p^2tj}$ and $a^{ptj}b^{-m}b^{-p^2tj}=a$. We know that $ba=ab^{p+1}$, so we can conclude that $ba^{pt}b^{-m}b^{-p^2tj}=$ $a^{ptj}b^{-m}b^{-p^2tj}b^{p+1}$, and so $a^{ptj}b^{(p+1)^{ptj}}b^{-m}=a^{ptj}b^{-m}b^{p+1}$ and $b^{(p+1)^{ptj}}=b^{p+1}$. But $b^{(p+1)^{ptj}}=b^{p^2tj+1}$, therefore we have $b^{p^2tj+1}=b^{p+1}$ and $b^{p^2tj-p}=1$ which implies that $p^3 \mid p^2tj-p$, a contradiction. Hence $D_\alpha(G)<G$ and G is not α-perfect. ■

In [4], has been shown that for every finite abelian group G, there exists a finite abelian group H and $\alpha \in Aut(H)$ such that $D_\alpha(H)\cong G$. Here, we may improve this result to finitely generated abelian groups.

Proposition 2.13. If $\alpha \in Aut(G)$ and $\beta \in Aut(H)$, then $D_{\alpha\times\beta}(G\times H)=$ $D_\alpha(G)\times D_\beta(H)$.

Proof. It is straightforward. ■

Theorem 2.14. Assume that $G=\underbrace{\mathbb{Z}\times\mathbb{Z}\times\cdots\times\mathbb{Z}}_{t-times}\times G_1$ such that G_1 is a finite abelian group. Then there exist an abelian group H and $\alpha \in Aut(H)$ such that $D_\alpha(H)\cong G$.

Proof. By Theorem 3.7 of [4], for finite group G_1, there exist abelian group H_1 and $\beta \in Aut(H_1)$ such that $D_\beta(H_1)\cong G_1$. Now, if $\alpha_i \in Aut(\mathbb{Z}_i\times\mathbb{Z}_i)$ by argument $(a,b)^{\alpha_i}=(a+b,b)$, then $D_{\alpha_i}(\mathbb{Z}_i\times\mathbb{Z}_i)=\langle -(a,b)+(a,b)^\beta : a,b\in\mathbb{Z}_i\rangle=\langle (b,0): b\in\mathbb{Z}_i\rangle\cong\mathbb{Z}$, where $\mathbb{Z}_i\cong\mathbb{Z}$ for $i=1,\dots,t$. Now, it is enough to put $H=\underbrace{\mathbb{Z}\times\mathbb{Z}\times\cdots\times\mathbb{Z}}_{2t-times}\times H_1$ and $\alpha=\alpha_1\times\cdots\times\alpha_t\times\beta$, then $D_\alpha(H)=D_{\alpha_1}(\mathbb{Z}\times\mathbb{Z})$ $\times D_{\alpha_2}(\mathbb{Z}\times\mathbb{Z})\times\cdots\times D_{\alpha_t}(\mathbb{Z}\times\mathbb{Z})\times D_\beta(H_1)\cong G$, and the proof is completed. ■

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