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Bounds of Adj-TVaR Prediction for Aggregate Risk

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Abstract

In financial and insurance industries, risks may come from several sources. It is therefore important to predict future risk by using the concept of aggregate risk. Risk measure prediction plays important role in allocating capital as well as in controlling (and avoiding) worse risk. In this paper, we consider several risk measures such as Value-at-Risk (VaR), Tail VaR (TVaR) and its extension namely Adjusted TVaR (Adj-TVaR). Specifically, we perform an upper bound for such risk measure applied for aggregate risk models. The concept and property of comonotonicity and convex order are utilized to obtain such upper bound.
Keywords: coherent property; comonotonic rv; convex order; tail property; Value-at-Risk (VaR).

Abstrak

Dalam industri keuangan dan asuransi, risiko dapat berasal dari beberapa sumber. Konsep risiko agregat diperlukan untuk memprediksi risiko di masa yang akan datang dari risiko tersebut. Prediksi ukuran risiko memiliki peran yang penting dalam mengalokasikan modal dan dalam mengendalikan atau menghindari risiko yang mungkin terjadi. Dalam artikel ini digunakan beberapa ukuran risiko seperti Value-at-Risk (VaR), Tail-VaR (TVaR), dan pengembangannya yaitu AdjustedTVaR (Adj-TVaR). Secara khusus, dihitung batas atas dari ukuran risiko pada model risiko agregat. Konsep dan sifat-sifat dari peubah acak komonotonik dan orde konveks digunakan dalam menentukan batas atas tersebut.
Kata kunci: ekor distribusi; orde konveks; peubah acak komonotonik; sifat koheren; *Value-at-Risk* (VaR).

1. INTRODUCTION

In financial and insurance industries, risks may come from several sources. It is therefore important to predict future risk by using the concept of aggregate risk. Suppose that S_N represents aggregate risk of collection of random losses $\{X_i : i = 1, 2, \ldots, N\}$ given by

$$S_N = X_1 + X_2 + \cdots + X_N ,$$

where the random losses are not necessarily independent, e.g. McNeil et al. [1], Tse [2]. Note that the random variable N is usually assumed to follow a discrete distribution whilst X_i is continuous random loss.

For a single random loss, X, with probability distribution determined by parameter vector θ, the risk measure Value-at-Risk (VaR) is defined as maximum tolerated risk at a given level of significance, e.g. McNeil et al. [1], Nieto and Ruiz [3]. VaR is very important in allocating capital as well as in controlling (and avoiding) worse risk. Basically, VaR is calculated via its inverse of distribution function i.e.

$$VaR_\alpha(X) = F_x^{-1}(\alpha),$$

for $a \in (0, 1)$. When we deal with either such inverse does not exist or the case of discrete loss, VaR may be obtained through $\inf\{x | F_X(x) \geq a\}$. Although VaR is widely used, it is not a coherent risk measure.

There have been some works carried out by authors to seek an improvement of VaR, besides describing formulas of VaR and CoVaR e.g. Nadarajah et al. [4]. Their efforts may be derived in two different directions. The first is improvement of VaR prediction accuracy i.e. the coverage probability of VaR prediction is closer to the target nominal or significant level. The example of this is an improved VaR in which the method was developed by Kabaila and Syuhada [5] [6]. The second improvement is seeking alternative risk measure that captures coherent property.

Provided VaR, we are able to calculate another risk measure as the mean of losses beyond VaR, known as Tail Value-at-Risk (TVaR), e.g. Artzner et al. [7], as follows

$$TVaR_\alpha(X) = E_\alpha[X | X > VaR_\alpha(X)] = \frac{1}{1-\alpha}\int_\alpha^1 VaR_u(X)\,du. \tag{1}$$

The value of TVaR in general is greater than its VaR and. Furthermore, TVaR is a coherent risk measure. On the extensions of TVaR is Modified CoVaR of Jadhav et al. [8] in which this proposed paper relies on loosely. In particular, our aim is to find an upper bound for TVaR applied for aggregate risk model. To do this, the concept of comonotonicity and convex order are utilized. Note that, although VaR is widely-used, it is not a coherent risk measure. It is therefore we apply and seek alternative risk measures.

This paper is organized as follows. Section 2 describes of Adjusted TVaR (Adj- TVaR) of Jadhav et al. [8]. The upper bound of Adj-TVaR for aggregate risk and its comonotonic counterpart is given in Section 3. Section 4 concludes.

2. THE RISK MEASURE OF ADJUSTED TAIL VALUE-AT-RISK (ADJ-TVaR)

The risk measure TVaR may be interpreted as the second tolerated risk after VaR. TVaR is expected to occur for less than $1 - a$. We may differentiate the value of TVaR for class of distributions, such as normal, heavy-tailed and extreme. The heavier of tail distribution the greater value of TVaR.

Theorem 2.1. (Jadhav et al. [8]) For a random loss X and some $\alpha \in (0,1)$ and $c \in [0,0.1]$, $Adj\text{-}TVaR_{(a,c)}(X)$ is defined as the mean loss in the interval between $VaR_\alpha(X)$ and $VaR_{\alpha+(1-\alpha)^{1+c}}(X)$ i.e.

$$Adj-TVaR_{(\alpha,c)}(X) = E[X | VaR_\alpha(X) \leq X \leq VaR_{\alpha+(1-\alpha)^{1+c}}(X) \tag{2}$$

where its confidence level is $\alpha + (1 - \alpha)^{1+c}$.

Let $VaR_a(X) = a$ and $VaR_{\alpha+(1-\alpha)^{1+c}}(X) = b$. From (2) we obtain

$$Adj-TVaR_{(\alpha,c)}(X) = \frac{1}{P(a \leq X \leq b)}\int_a^b x f_X(x)\,dx = \frac{1}{(1-\alpha)^{1+c}}\int_a^b x f_X(x)\,dx.$$

By substituting $F_X(x) = \mu$, $x = F^{-1}(\mu)$, and $f(x)dx = d\mu$,

$$Adj-TVaR_{(\alpha,c)}(X) = \frac{1}{(1-\alpha)^{1+c}} \int_{\alpha}^{\alpha+(1-\alpha)^{1+c}} F_X^{-1}(\mu)\,\mathrm{d}\,\mu.$$

For the case of an aggregate risk, S_N, the Adj-TVaR$_{(a,c)}(S_N)$ is given by

$$Adj-TVaR_{(\alpha,c)}(S_N) = \frac{1}{(1-\alpha)^{1+c}} \int_{\alpha}^{\alpha+(1-\alpha)^{1+c}} F_{S_N}^{-1}(\mu)\,\mathrm{d}\,\mu.$$

PROPERTY-2.1. The Adj-TVaR is a coherent risk measure.

To verify that Adj-TVaR is coherent risk measure, it will be shown that it fulfill subadditivity properties [5]. Consider X_1 and X_2 be individual risk and S_2 aggregate of X_1 and X_2. Thus,

$$\begin{aligned}
&(1-\alpha)^{1+c}(\text{Adj-TVaR}_{(\alpha,c)}(X_1)+\text{Adj-TVaR}_{(\alpha,c)}(X_2)-\text{Adj-TVaR}_{(\alpha,c)}(S_2))\\
&=(1-\alpha)^{1+c}\Big(E\Big[X_1 \mid F_{X_1}^{-1}(\alpha)\le X_1\le F_{X_1}^{-1}\big(\alpha+(1-\alpha)^{1+c}\big)\Big]+\\
&\quad E\Big[X_2 \mid F_{X_2}^{-1}(\alpha)\le X_2\le F_{X_2}^{-1}\big(\alpha+(1-\alpha)^{1+c}\big)\Big]\Big)-(1-\alpha)^{1+c}\\
&\quad \Big(E\Big[S_2 \mid F_{S_2}^{-1}(\alpha)\le S_2\le F_{S_2}^{-1}\big(\alpha+(1-\alpha)^{1+c}\big)\Big]\Big)\\
&=E\Big[X_1\mathbf{I}_{F_{X_1}^{-1}(\alpha)\le X_1\le F_{X_1}^{-1}(\alpha+(1-\alpha)^{1+c})}\Big]+E\Big[X_2\mathbf{I}_{F_{X_2}^{-1}(\alpha)\le X_2\le F_{X_2}^{-1}(\alpha+(1-\alpha)^{1+c})}\Big]-\\
&\quad E\Big[S_2\mathbf{I}_{F_{S_2}^{-1}(\alpha)\le S_2\le F_{S_2}^{-1}(\alpha+(1-\alpha)^{1+c})}\Big]\\
&=E\Big[X_1\Big(\mathbf{I}_{F_{X_1}^{-1}(\alpha)\le X_1\le F_{X_1}^{-1}(\alpha+(1-\alpha)^{1+c})}-\mathbf{I}_{F_{S_2}^{-1}(\alpha)\le S_2\le F_{S_2}^{-1}(\alpha+(1-\alpha)^{1+c})}\Big)\Big]+\\
&\quad E\Big[X_2\Big(\mathbf{I}_{F_{X_2}^{-1}(\alpha)\le X_2\le F_{X_2}^{-1}(\alpha+(1-\alpha)^{1+c})}-\mathbf{I}_{F_{S_2}^{-1}(\alpha)\le S_2\le F_{S_2}^{-1}(\alpha+(1-\alpha)^{1+c})}\Big)\Big]
\end{aligned}$$

and we obtain

$$\begin{aligned}
&(1-\alpha)^{1+c}(\text{Adj-TVaR}_{(\alpha,c)}(X_1)+\text{Adj-TVaR}_{(\alpha,c)}(X_2)-\text{Adj-TVaR}_{(\alpha,c)}(S_2))\\
&\ge F_{X_1}^{-1}\big(\alpha+(1-\alpha)^{1+c}\big)E\Big[X_1\mathbf{I}_{F_{X_1}^{-1}(\alpha)\le X_1\le F_{X_1}^{-1}(\alpha+(1-\alpha)^{1+c})}-\\
&\quad \mathbf{I}_{F_{S_2}^{-1}(\alpha)\le S_2\le F_{S_2}^{-1}(\alpha+(1-\alpha)^{1+c})}\Big]+F_{X_2}^{-1}(\alpha)\big(\alpha+(1-\alpha)^{1+c}\big)\\
&\quad E\Big[\mathbf{I}_{F_{X_2}^{-1}(\alpha)\le X_2\le F_{X_2}^{-1}(\alpha+(1-\alpha)^{1+c})}-\mathbf{I}_{F_{S_2}^{-1}(\alpha)\le S_2\le F_{S_2}^{-1}(\alpha+(1-\alpha)^{1+c})}\Big]\\
&=F_{X_1}^{-1}\big(\alpha+(1-\alpha)^{1+c}\big)\Big(\big(\alpha+(1-\alpha)^{1+c}-\alpha\big)-\big(\alpha+(1-\alpha)^{1+c}-\alpha\big)\Big)+\\
&\quad F_{X_2}^{-1}\big(\alpha+(1-\alpha)^{1+c}\big)\Big(\big(\alpha+(1-\alpha)^{1+c}-\alpha\big)-\big(\alpha+(1-\alpha)^{1+c}-\alpha\big)\Big)\\
&=0
\end{aligned}$$

Thus, for $S_2 = X_1 + X_2$, we have

$$\text{Adj-TVaR}_{(\alpha,c)}(S_2) \le \text{Adj-TVaR}_{(\alpha,c)}(X_1) + \text{Adj-TVaR}_{(\alpha,c)}(X_2)$$

that proves subadditivity property.

The following figures illustrate the value of Adj-TVaR for several distribution on the probability function curve, in comparison to its VaR.

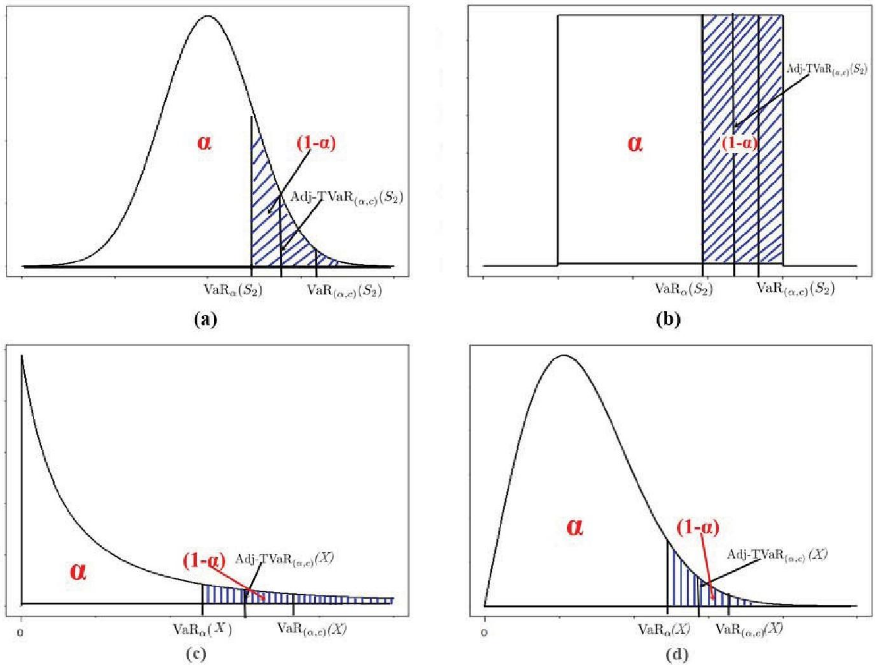

Figure 1: Adj-TVaR$_{\alpha}(X)$ on probability function curves: (a) Normal, (b) Uniform, (c) Pareto, (d) Weibull

PrOPERTY-2.2. The value Adj-TVaR, in the same level of confidence, has lower value than its corresponding TVaR.

To prove the above property, we will show that at $a + (1 - a)^{1+c}$ significance level, $\text{TVaR}_{\alpha+(1-\alpha)^{1+c}}(X) \geq \text{Adj-TVaR}_{(\alpha,c)}(X)$. Suppose that on right-tailed there is two intervals, $I_1 = \left[\alpha, 1-\alpha^{1+c}\right]$ and $I_2 = \left[\alpha+(1-\alpha)^{1+c}, 1\right]$. Thus, for every $p \in I_1$ and $q \in I_2$, $F_X^{-1}(p) = F_X^{-1}(q)$, such that

Let $1 - (1 - a)^{1+c} = a$ and $a + (1 - a)^{1+c} = b$, we have

$$\int_a^b F_X^{-1} r\, dr + \int_b^1 F_X^{-1} p\, dp \geq \int_\alpha^a F_X^{-1} q\, dq + \int_a^b F_X^{-1} r\, dr.$$

$$\int_b^1 F_X^{-1} p\, dp \geq \int_\alpha^a F_X^{-1} q\, dq$$

from which we can conclude that $\text{TVaR}_{\alpha+(1-\alpha)^{1+c}}(X) \geq \text{Adj-TVaR}_{(\alpha,c)}(X)$.

3. UPPER BOUND OF ADJ-TVaR FOR AGGREGATE RISK

When a risk measure is applied to aggregate risk model, it may not be easy to the equality between the risk measure of aggregate risk and the aggregate of risk measure of individual risk. In other words, we may only seek whether subadditivity property applies for such risk measure.

As before, consider an aggregate risk of S_N. Let $\mathbf{X} = (X_1, X_2, ..., X_N)$ be random vector and

$F_{X_1}, F_{X_2}, F_{X_3}, \ldots, F_{X_N}$ are their corresponding marginal distribution function. Now, for any random vector **X** not necessarily comonotonic, its comonotonic counter- part is defined as any random vector with the same marginal distributions and with the comonotonic dependency structure. It can be proven that a random vector is comonotonic if and only if all its components are non-decreasing (or non-increasing) functions of the same random losses (see McNeil et al., 2005).

The comonotonic counterpart of **X** will be denoted by $\mathbf{X}^c = (X_1{}^c, X_2{}^c, \ldots, X_N{}^c)$ and

$$(X_1{}^c, X_2{}^c, \ldots, X_N{}^c) \stackrel{d}{=} (F_{X_1}^{-1}(U), F_{X_2}^{-1}(U), \ldots, F_{X_N}^{-1}(U)).$$

A random vector is comonotonic if and only if its marginal distribution function are non-decreasing function of the same random variable. Suppose that S^c be the sum of every component of X^c. S^c can be expressed as

$$S_N{}^c = X_1{}^c + X_2{}^c + \ldots + X_N{}^c.$$

Definition 3.1. Let X_1 and X_2 be two random variables.

(i) X_1 is said to be smaller than X_2 in convex order ($X_1 \leq_{co} X_2$) if

$$E\left[g(X_1)\right] \leq E\left[g(X_2)\right]$$

for any convex function g such that the expectation exist.

(ii) X_1 is said to be smaller than X_2 in stop loss order ($X_1 \leq_{sl} X_2$) if

$$E\left[(X_1 - K)_+\right] \leq \left[E(X_2 - K)_+\right] and\ E[X_1] = E[X_2],$$

for every $K \in R$.

PrOPERTY-3.1. If X_1 precedes X_2 in convex order sense i.e if $X_1 \leq_{co} X_2$, then $E[X_1] = E[X_2]$ and $\mathrm{Var}[X_1] \leq \mathrm{Var}[X_2]$.

PrOPERTY-3.2. $X_1 \leq_{co} X_2$ if only if $X_1 \leq_{sl} X_2$.

Theorem 3.1. For any random vector $\mathbf{X} = (X_1, X_2, \ldots, X_N)$ we have that [9]

$$S_N \leq_{co} S_N{}^c \tag{3}$$

In the following proposition, we argue that risk measure of Adj-TVaR for aggregate risk is lower than the corresponding Adj-TVaR for its comonotonic counterpart.

Proposition 3.1. For any aggregate random variable S_N and S_N^c counterpart of it, we have that if $S_N \leq_{sl} S_N^c$ then their respective Adj-TVaR are ordered:

$$S_N \leq_{sl} S_N^c \Rightarrow Adj - TVaR_{(\alpha,c)}(S_N) \leq Adj - TVaR_{(\alpha,c)}(S_N^c) \tag{4}$$

Proof: First, we assume $S_N \leq_{sl} S_N^c$. According to Dhaene et al. [10] if $S_N \leq_{sl} S_N^c$ than $\mathrm{VarR}_\alpha\left(S_N\right) \leq \mathrm{VarR}_\alpha\left(S_N^c\right)$. That inequality causes

$$\begin{aligned}\text{Adj-TVaR}_{\alpha,c}(S_N) &= E\left[S_N \mid \mathrm{VaR}_\alpha(S_N) \leq S_N \leq \mathrm{VaR}_{\alpha+(1-\alpha)^{1+c}}(S_N)\right] \\ &\leq E\left[S_N^c \mid \mathrm{VaR}_\alpha(S_N^c) \leq S_N^c \leq \mathrm{VaR}_{\alpha+(1-\alpha)^{1+c}}(S_N^c)\right] \\ &= \text{Adj-TVaR}_{\alpha,c}(S_N^c).\end{aligned}$$

The formula for $\mathrm{Adj}-\mathrm{TVaR}_{\alpha,c}(S_N^c)$ is defined as

$$\text{Adj-TVaR}_{\alpha,c}(S_N^c) = E\left[S_N^c \mid \mathrm{VaR}_\alpha(S_N^c) \leq S_N^c \leq \mathrm{VaR}_{\alpha+(1-\alpha)^{1+c}}(S_N^c)\right]$$

Let $\mathrm{VaR}_\alpha(S_N^c) = a$ and $\mathrm{VaR}_{\alpha+(1-\alpha)^{1+c}}(S_N^c) = \mathrm{b}$. Thus

$$\begin{aligned}\text{Adj-TVaR}_{\alpha,c}(S_N^c) &= \frac{1}{P(a \leq S_N^c \leq b)} \int_a^b s f_{S_N^c}(s) ds \\ &= \frac{1}{(1-\alpha)^{1+c}} \int_a^b s f_{S_N^c}(s) ds.\end{aligned}$$

By substituting $F_{S_N^c}(s) = \mu, s = F_{S_N^c}^{1}(\mu)$ and $f_{S_N^c}(s)ds = d\mu,$ we obtain

$$\text{Adj-TVaR}_{\alpha,c}(S_N^c) = \frac{1}{(1-\alpha)^{1+c}} \int_\alpha^{\alpha+(1-\alpha)^{1+c}} F_{S_N^c}^{-1}(\mu) d\mu.$$

Suppose that $\alpha + (1-\alpha)^{1+c} = b,$ we have the following

$$\begin{aligned}\text{Adj-TVaR}_{\alpha,c}(S_N^c) &= \frac{1}{(1-\alpha)^{1+c}}\left(\int_\alpha^b F_{X_1}^{-1}(\mu) d\mu + \ldots + \int_\alpha^b F_{X_N}^{-1}(\mu) d\mu\right) \\ &= \text{Adj-TVaR}_{(\alpha,c)}(X_1) + \ldots + \text{Adj-TVaR}_{(\alpha,c)}(X_N),\end{aligned}$$

and this proves our proposition.

4. CONCLUDING REMARK

The risk measure of Adj-TVaR for aggregate risk and its comonotonic counter- part may be applied to the Copula TVaR of Brahim et al. [11]. In practice, many random losses really depend on other losses that are not necessarily its component of aggregate risk.

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