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Rainbow Connection Number on Amalgamation of General Prism Graph

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Abstract

Let G be a nontrivial connected graph, the rainbow-k-coloring of graph G is the mapping of $c: E(G) \rightarrow \{1,2,3,\cdots,k\}$ such that any two vertices from the graph can be connected by a rainbow path (the path with all edges of different colors). The least natural number k of k-edge coloring graph G is called the rainbow connection number of G, denoted by rc(G). For $t \in N$ and $t \geq 2$, let $\{P_{(m,2)i} | i \in (1,2,\cdots,t)$, $m \geq 3\}$ is a collection of prism graph with a fixed vertex v called terminal. The amalgamation of such a prism graph is denoted by $Amal_t(P_{m,2})$ with $m \geq 3$. We figured that $rc(Amal_t(P_{m,2}))$ forms a particular pattern and this research aims to find the formula of such pattern. The result of this research is $\left(Amal_t(P_{m,2})\right) = 2\left\lceil\frac{m+1}{2}\right\rceil, t \geq 2$ and $m \geq 3$.

Keywords: rainbow coloring; amalgamation of graph; prism graph.

Abstrak

Misal G adalah graf terhubung non trivial, pewarnaan pelangi-k dari graf G adalah pemetaan $c: E(G) \rightarrow \{1,2,3,\cdots,k\}$ sedemikian sehingga setiap dua titik dari graf dapat dihubungkan dengan lintasan pelangi (lintasan dengan semua sisi yang berbeda warna). Bilangan asli terkecil k dari k-sisi pewarnaan graf G demikian sehingga G memiliki rainbow-k-coloring merupakan rainbow connection number bagi G, dinotasikan dengan $rc(G)$. Untuk $t \in N$ and $t \geq 2$, maka $\{P_{(m,2)i} | i \in (1, 2,.....,t\}$, $m \geq 3\}$ adalah kumpulan graf prisma yang memiliki titik tertentu v yg disebut terminal. Amalgamasi graf prisma biasa dinotasikan dengan $Amal_t(P_{m,2})$ dengan $m \geq 3$. Jika digambarkan $rc(Amal_t(P_{m,2}))$ akan membentuk suatu pola dan penelitian ini bertujuan untuk menemukan pola dari graf tersebut. Hasil dari penelitian ini adalah $\left(Amal_t(P_{m,2})\right) = 2\left\lceil\frac{m+1}{2}\right\rceil, t \geq 2$ dan $m \geq 3$.

Kata kunci: pewarnaan pelangi, amalgamasi graf, graf prisma.

1. INTRODUCTION

The basic idea of graph were first introduced by Leonhard Euler when he attempted to find the solution for the Königsberg bridge problem [1]. The one branch of graph is graph labeling that introduced by Rosa [2] and graph coloring is a special case of this graph. Graph coloring has been one of the more popular branches in mathematics. The rainbow connection number is one of the colorings of graph theory that is interesting to talk about, rainbow connection number is a unique type of coloring because it requires different colors to color the side of the graph. Rainbow connection number can also be considered as something that has been discussed quite often, proven by the number of journals that have talked about this topic. Chartrand et al. [3] first introduced a rainbow connection number.

Let G be a nontrivial connected graph where we define a coloring $c: E(G) \to \{1,2,\ldots,k\}, k \in \mathbb{N}$ where adjacent edges in G may have the same color. Such coloring is called rainbow coloring in graph G if graph G is rainbow-connected, that is, when any two vertices u, v in G contains a uv rainbow path-a path that consists no two edges with the same color. If k color is used, then the coloring c of graph G is said to be a rainbow k-coloring with k being its rainbow connection number in graph G, denoted as $rc(G)$ [4].

Charttrand et al. [3] determined rainbow connection number in some special types of graphs such as the wheel graph, the k-partite complete graph, the tree graph, and the cycle graph. Some research in rainbow connection number is Kumala and Salman [4] discuss about the rainbow connection number for the flower graph (C_m, K_n), Darmawan [5] discussed about the rainbow connection number on some special graphs, and Palupi et al. [6] determined the rainbow connection number in prism graph $P_{3,2}$ amalgamation which resulted $rc(G) = 4$. Similar to Palupi et al. [6], we will explore rainbow connection number in prism graph $P_{m,2}$ amalgamation to obtain a more general theorem.

2. PRISM GRAPH

Prism Graph is the Cartesian product $C_m \times P_n$, that is, a cycle graph with m vertices and a path with n vertices, denoted by $P_{m,n}$. The set $V(P_{m,n}) = \{v_{i,j} : 1 \leq i \leq m, 1 \leq j \leq n\}$ and $E(P_{m,n}) = \{v_{i,j}\, v_{i+1,j} : 1 \leq i \leq m-1, 1 \leq j \leq n\} \cup \{v_{m,j}\, v_{1,j} : 1 \leq j \leq n\} \cup \{v_{i,j}\, v_{i,j+1} : 1 \leq i \leq m, 1 \leq j \leq n-1\}$ act as its vertices and edges set respectively [6]. Fitriani and Salman [7] show that the lower bound and upper bound of a connection for any graph. They proved that the lower limit of the number of rainbow connection in graphs is the diameter of the graph itself. Figure 1 illustrates the Prism Graph $P_{3,2}$.

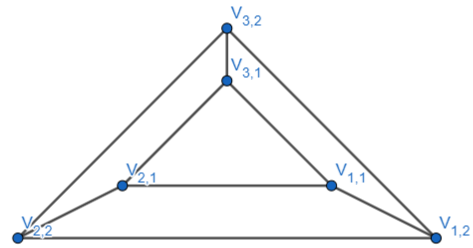

Figure 1. Prism graph $P_{3,2}$.

Theorem 1. For $t \in N$, $t \geq 2$, in the case of $\{G_i, i \in 1, 2, \ldots, t\}$ being a group of finite graphs and each G_i has fixed point v_{0i} called terminal. If G is the amalgamation of $G_1, G_2, \ldots, G_t$, amal (G_i, v_{01}), then

$$\mathrm{diam}(G) \leq \mathrm{rc}(G) \leq \sum_{i=1}^{t} \mathrm{rc}(G_i). \quad (1)$$

3. RESULT AND DISCUSSION

We begin to pay attention to the diameter of a circle graph C_m. Figure 2 illustrates that the diameter of graph C_m with $m \geq 3$ will never exceed half of the circumference in its cycle and $\left\lceil \frac{m-1}{2} \right\rceil$ to be exact.

Lemma 2. If graph $P_{m,2}$ is a prism graph with $m \geq 3$ then $diam(P_{m,2}) = \left\lceil \frac{m+1}{2} \right\rceil$.

Proof. Graph $P_{m,2}$ is the product of $C_m \times P_2$, therefore, its figure will form a double-layered Graph C with two layers connected. See Figure 3. From this figure, the diameter of the prism graph is a distance from one point of the outer graph C_m to the inner graph C_m. Hence it needs an additional one side from $diam(C_m)$. Therefore we can write:

$$diam(P_{m,2}) = \lceil diam\ (C_m) + 1 \rceil = \left\lceil \frac{m-1}{2} + 1 \right\rceil = \left\lceil \frac{m+1}{2} \right\rceil. \blacksquare$$

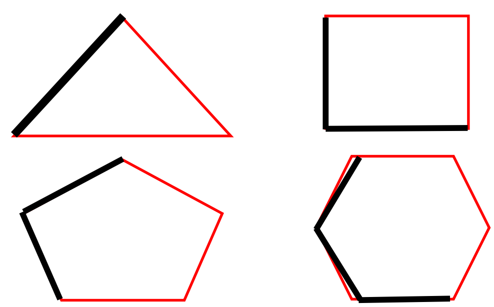

Figure 2. Illustration for $\boldsymbol{diam(C_m)}$.

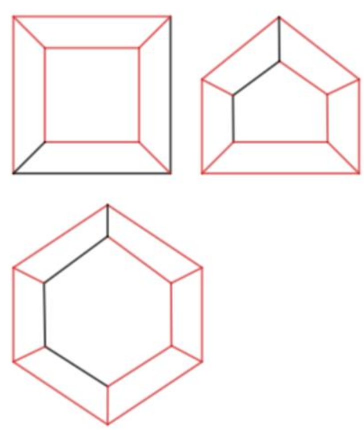

Figure 3. Graph prism diameter.

Lemma 3. If graph $Amal_t(P_{m,2})$ is the amalgamation of prism graph with $m \geq 3$ then $\text{diam}(Amal_t(P_{m,2})) = 2\left\lceil \frac{m+1}{2} \right\rceil$.

Proof. As we know, $Amal_t(P_{m,2})$ is a group of a minimum of 2 prism graphs connected by a terminal vertex. Because the longest distance of any two points in a said graph has to go pass its terminal vertex as illustrated in Figure 4, then the diameter is twice of the diameter of a prism graph, hence the diameter of this graph is $2 \times diam(P_{m,2})$. □

3.1. Rainbow Connection Number on Prism Graph $\mathbf{P_{(3,2)}}$ Amalgamation

See Picture 5. From this figure, we achieve that $rc\left(Amal_t(P_{(3,2)})\right) = 4$. Note that the value of t will not affect the value of the rainbow connection number because every path will pass the terminal point.

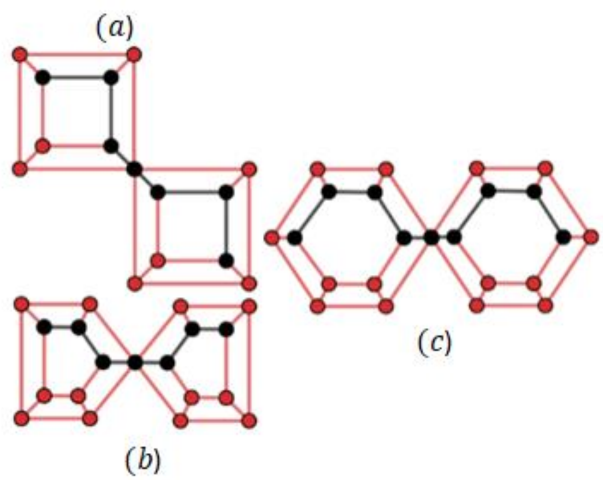

Figure 4. Diameter of graph $\boldsymbol{Amal_t(P_{m,2})}$), for $\boldsymbol{t = 2}$.

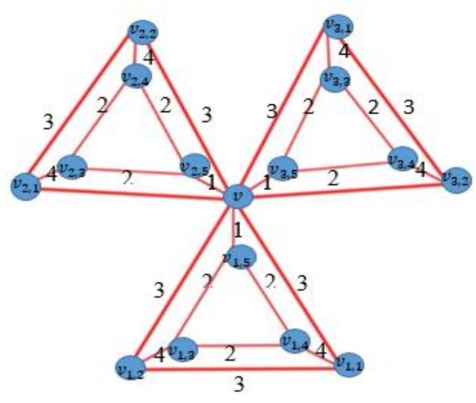

Figure 5. $\boldsymbol{rc(Amal_t(P_{3,2})}$), for $\boldsymbol{t = 3}$.

3.2. Rainbow Connection Number$\left(\mathbf{Amal_t(P_m, 2)}\right)$

The next theorem is the result of the coloring of rainbow connection number in prism graph $\boldsymbol{P_{m,2}}$ amalgamations.

Theorem 4. If $t \geq 2$ and $G \cong Amal(G_i, v)$ for every $i \in \{1,2,3,\dots t\}$, with G_i is prism graph $P_{m,2}$ with $m \geq 3$, then

$$rc(G) = 2\left\lceil \frac{m+1}{2} \right\rceil.$$

Proof. It will be shown that there is a rainbow connection number with a value of $2\left\lceil \frac{m+1}{2} \right\rceil$. Because $\left(Amal_t(P_{m,2})\right) = 2\left\lceil \frac{m+1}{2} \right\rceil$, then according to Theorem 1 we must have $rc\left(Amal_t(P_{(m,2)})\right) = 2\left\lceil \frac{m+1}{2} \right\rceil$.

Define the following proposition function:

$$P(m):\ rc\big(Amal_t(P_{m,2})\big) = 2\left\lceil \frac{m+1}{2} \right\rceil, m, t \in N,\ m \geq 3.$$

It will be shown that $P(m)$ is true for every $m \in \mathbb{N}$ with $m \geq 3$ via the principle of mathematical induction.

(i) It will be shown that $P(3)$ is true
According to Subsection 3.2., it is known that $rc(Amal_t(P_{3,2})) = 4$. See that

$$rc\left(Amal_t(P_{3,2})\right) = 2\left\lceil \frac{3+1}{2} \right\rceil = 2\left\lceil \frac{4}{2} \right\rceil = 2\lceil 2 \rceil = 4.$$

Therefore, $P(3)$ is true.

(ii) Assume that $P(m)$ is true.

(iii) It will be shown that $P(m+1)$ is true.
Because $P(m)$ is true then it can be inferred that $rc\big(Amal_t(P_{m,2})\big) = 2\left\lceil \frac{m+1}{2} \right\rceil$.

Case 1. $m = 2p$ for an arbitrary $p \in N$, $p \geq 2$

We know that $rc(Amal_t(P_{2p,2})) = 2\left\lceil \frac{2p+1}{2} \right\rceil = 2p+2$, thus we will try to show that $rc(Amal_t(P_{2p+1,2})) = 2\left\lceil \frac{2p+1+1}{2} \right\rceil = 2p+2$.

For example, Figure 6 is a pair of the farthest side from v. From this illustration, it is clear that any additional color is not needed when we add an extra layer. Moreover, if we look at it from the newly formed points in $Amal_t(P_{2p,2})$ tracks to the already existing points can be found without repeating a color, thus

$$rc(Amal_t(P_{2p+1,2})) = 2\left\lceil \frac{2p+1+1}{2} \right\rceil = 2p+2.$$

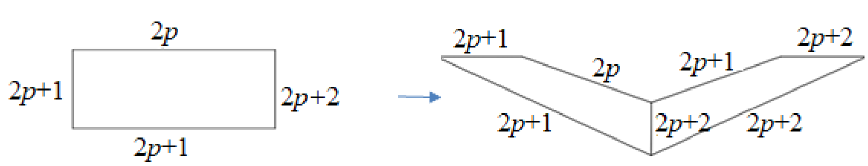

Figure 6. Change from even side to odd side.

Case 2. $m = (2p-1)$ for an arbitrary $p \in \mathbb{N}$, $p \geq 3$ then we know that $rc\left(Amal_t\left(P_{(2p-1),2}\right)\right) = 2\left\lceil\frac{(2p-1)+1}{2}\right\rceil = 2p$. We will then show that $rc(Amal_t(P_{2p,2})) = 2\left\lceil\frac{2p+1}{2}\right\rceil = 2p+2$.

For example, Figure 7 is a pair of the farthest side from v. From this illustration, clearly that 2 additional colors will be needed and if we look at it from the newly formed points in $Amal_t(P_{2p,2})$ a track to the already existing points can be found without repeating a color, thus it has to be

$$rc(Amal_t(P_{2p,2})) = 2\left\lceil\frac{2p+1}{2}\right\rceil = 2p+2.$$

According to (i), (ii), and (iii) and the principle of mathematical induction then we may conclude that

$$rc\left(Amal_t(P_{m,2})\right) = 2\left\lceil\frac{m+1}{2}\right\rceil,\ m, t \in N, m \geq 3. \blacksquare$$

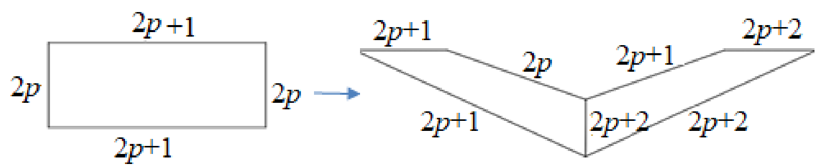

Figure 7. Change from odd sides to even sides.

From Lemma 3 we got $diam\ Amal_t\left(P_{m,2}\right) = 2\left\lceil\frac{m+1}{2}\right\rceil$, $m,t \in N, m \geq 3$ and from Theorem 4 we got $rc\left(Amal_t(P_{m,2})\right) = 2\left\lceil\frac{m+1}{2}\right\rceil$, $m,t \in N, m \geq 3$.

Result Illustrations for $m = 4, 5, 6, 7$.

Rainbow connection number on prism graph $P_{(4,2)}$ amalgamations. From Figure 8 we have $rc\left(Amal_t\left(P_{(4,2)}\right)\right) = 6$. Rainbow Connection Number on prism graph $P_{(5,2)}$ amalgamations. From Figure 9 we have $rc\left(Amal_t\left(P_{(5,2)}\right)\right) = 6$. Rainbow Connection Number for prism graph $P_{(6,2)}$ amalgamations. From Figure 10 we have $rc\left(Amal_t\left(P_{(6,2)}\right)\right) = 8$. Rainbow Connection Number on prism graph $P_{(7,2)}$ amalgamations. From Figure 11 we have $rc\left(Amal_t\left(P_{(7,2)}\right)\right) = 8$. Hence, we get: $rc\left(G\left(P_{(3,2)}\right)\right) = 4$, $rc\left(G\left(P_{(4,2)}\right)\right) = 6$, $rc\left(G\left(P_{(5,2)}\right)\right) = 6$, $rc\left(G\left(P_{(6,2)}\right)\right) = 8$, $rc\left(G\left(P_{(7,2)}\right)\right) = 8$ which is so far are consistent to Theorem 4.

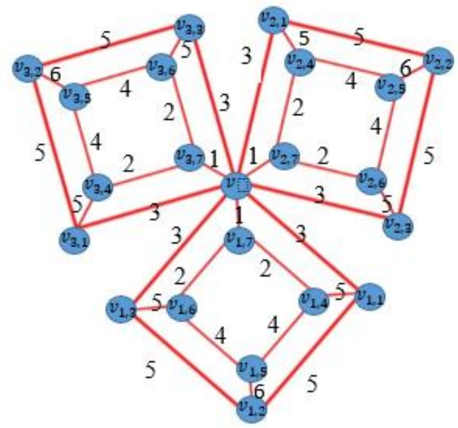

Figure 8. Rainbow coloring of $\boldsymbol{Amal_t(P_{4,2})}$, for $\boldsymbol{t = 3}$.

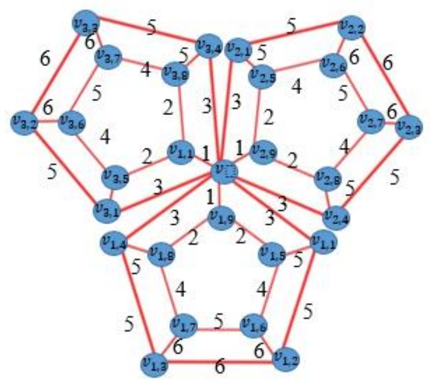

Figure 9. Rainbow coloring of $Amal_t(P_{5,2})$, for $\mathrm{t} = 3$.

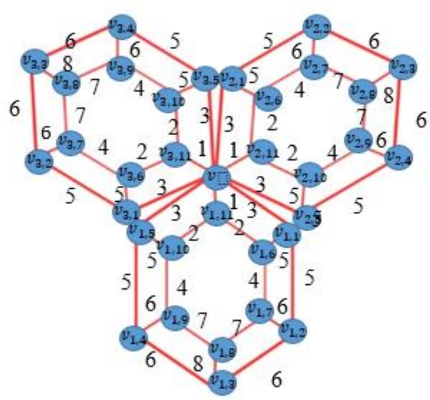

Figure 10. Rainbow coloring of $\boldsymbol{Amal_t(P_{6,2})}$, for $\mathbf{t = 3}$.

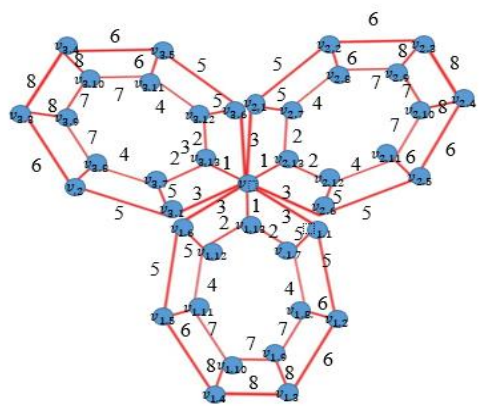

Figure 11. Rainbow coloring of $\boldsymbol{Amal_t(P_{7,2})}$, for $\boldsymbol{t = 3}$.

4. CONCLUSION

We get a general formula using the theorems that we got to obtain rainbow connection number on prism graph $P_{m,2}\ amalgamation$ is $rc\big(Amal_t(P_{m,2})\big) = 2\left\lceil\frac{m+1}{2}\right\rceil, m \geq 3, m \in N$ and the diameter on $Amal_t\big(P_{m,2}\big) = 2\left\lceil\frac{m+1}{2}\right\rceil, m \geq 3, m \in N$.

REFERENCES

[1] R. Trudeau, Introduction to Graph Theory, New York: Dover Publication, Inc., 1994.

[2] A. Rosa, "On Certain Valuations of the Vertices of a Graph," in *Theory of Graphs, Int. Symp. Rome July 1966*, Gordon and Breach, 1967.

[3] G. Chartrand, G. Johns, K. McKeon and P. Zhang, "Rainbow Connection in Graphs," *Math Bohem,* vol. 133, no. 1, pp. 85 - 98, 2008.

[4] I. Kumala and A. Salman, "The Rainbow Connection Number of a Flower (C_m,K_n) Graph and a Flower (C_3,F_n) Graph," *Procedia Computer Science,* vol. 74, pp. 168 - 172, 2015.

[5] R. Darmawan, "Analisis Rainbow Connection Number pada Graf Khusus dan Hasil Operasinya. Thesis," Pascasarjana Universitas Jember, 2015.

[6] C. Palupi, W. Aribowo, Y. Irene and I. Hasanah, "On Rainbow Connection and Strong Rainbow Connection Number from Amalgamation of Prism Graph P_3,2," *Journal Of Physics: Conference Series,* vol. 1008(1), 2018.

[7] D. Fitriani and A.N.M. Salman, "Rainbow Connection Number of Amalgamation of some Graphs," *AKCE International Journal of Graphs and Combinatorics ,* vol. 13, pp. 90 - 99, 2016.

[8] K. Sugeng, S. Surjadi and R. Denny, Teori Graf dan Aplikasinya, Depok: Department of Mathematics FMIPA University of Indonesia, 2014.

[9] K. Rosen, Discrete Mathematics and Its Applications, New York: Mac Graw-Hill, inc, 2012.

[10] N. Hartsfield and G. Ringel, Pearls in Graph Theory, New Yotk: Dover Publications, Inc, 1994.

[11] G. Chartrand and O. R. Oellermann, Applied and Algorithmic Graph Theory, New York: McGraw-Hill, 1993.