The radial wave function of 2D and 3D quantum harmonic oscillator

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Abstract. One dimensional quantum harmonic oscillator is well studied in elementary textbooks of quantum mechanics. The wave function of one-dimensional oscillator harmonic can be written in term of Hermite polynomial. Due to the symmetry of the spring energy, the wave functions of two-dimensional and three-dimensional harmonic oscillators can be written as products of the one-dimensional case. Because of that, the wave functions of two- and three-dimensional cases are focused on cartesian coordinates. In this article, we utilize polar and spherical coordinates to describe the wave function of two- and three-dimensional harmonic oscillators, respectively. The radial part of the wave functions can be written in term of associated Laguerre polynomials.

Keywords: associated Laguerre polynomials, quantum harmonic oscillator, radial wave function, two dimension quantum harmonic oscillator, three dimension quantum harmonic oscillator.

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INTRODUCTION

In elementary textbooks of quantum mechanics, one dimensional quantum harmonic oscillator is well studied using a potential energy analogous to classical harmonic oscillator

\[ V(x) = \frac{1}{2} M \omega^2 x^2. \]  

(1)

Here \( M \) is mass of the particle, \( \omega \) is the characteristic radial frequency. The eigen wave function of the Schrödinger equation \((p^2/(2m) + V(x))\psi = i\hbar \psi\) can be written in term of Hermite polynomials \( H_n \) \[1\]

\[ \psi_{1d} = e^{-\frac{1}{2} \alpha^2 x^2} \frac{\alpha}{\sqrt{2^n n! \sqrt{\pi}}} e^{-\frac{1}{2} \alpha^2 x^2} H_n(\alpha x). \]  

(2)

Here \( \alpha = \sqrt{\omega / \hbar} \), and \( E_n = \left(n + \frac{1}{2}\right) \hbar \omega \) is the quantized energy.

However, the potential is a central potential \( V(\vec{r}) = \frac{1}{2} m \omega^2 r^2 \) and therefore the eigen wave functions should be able to be written as a product of radial and angular wave functions \[2\]. The angular wave functions of two- and three-dimensional potentials with central symmetry are well-described using sinusoidal and spherical harmonics
functions, respectively [3]. However, studies of higher dimension quantum oscillators elude the discussion of radial wave function [4]–[8].

It may be due to the additive property of the norm of \( \vec{r} \). Because of that, the generalization of the potential energy into two and three dimensions is straightforward

\[
V(\vec{r}) = \begin{cases} \frac{1}{2} M \omega^2 (x^2 + y^2), & \text{for 2d}, \\ \frac{1}{2} M \omega^2 (x^2 + y^2 + z^2), & \text{for 3d}. \end{cases}
\]  

(3)

Because of this symmetry, the wave functions of two- and three-dimensional cases are focused on cartesian coordinates

\[
\psi_{2d} = e^{-i(n_x + n_y + 1)\omega t} \left( \frac{\alpha}{2^n n! \sqrt{\pi}} \right) e^{-\frac{1}{2} \alpha^2(x^2+y^2)} H_{n_x}(\alpha x) H_{n_y}(\alpha y),
\]

(4)

\[
\psi_{3d} = e^{-i(n_x + n_y + n_z + \frac{3}{2})\omega t} \left( \frac{\alpha}{2^n n! \sqrt{\pi}} \right)^3 e^{-\frac{1}{2} \alpha^2(x^2+y^2+z^2)} H_{n_x}(\alpha x) H_{n_y}(\alpha y) H_{n_z}(\alpha z).
\]

Coefficient of time-dependent exponential indicates that the energy eigenvalues also have the additive properties.

\[
E_n = \begin{cases} \hbar \omega(n_x + n_y + 1), & \text{for 2d}, \\ \hbar \omega(n_x + n_y + n_z + \frac{3}{2}), & \text{for 3d}. \end{cases}
\]  

(5)

In this article, we utilize polar and spherical coordinates to describe the wave function of two- and three-dimensional harmonic oscillators, respectively. The understanding of radial wave function should provide better insight on the mathematical physics of basic quantum mechanics.

**VARIABLES SEPARATIONS OF SCHRÖDINGER EQUATION**

Schrodinger equation of quantum particle under the influence of central potential energy

\[
\left(-\frac{\hbar^2}{2M} \nabla^2 + \frac{1}{2} m \omega^2 r^2 \right) \psi(\vec{r}, t) = i \hbar \frac{\partial}{\partial t} \psi(\vec{r}, t),
\]

(6)

can be rewritten using variable separation with energy’s eigen values \( E \)

\[
\frac{1}{u(\vec{r})} \left( -\frac{\hbar^2}{2M} \nabla^2 + \frac{1}{2} m \omega^2 r^2 \right) u(\vec{r}) = E = \frac{i \hbar}{T(t)} \frac{\partial T(t)}{\partial t},
\]

(7)

to arrive at the time-independent wave function \( u(\vec{r}) \) and space-independent \( T(t) \). One can show that straight-forwardly shows that

\[
T(t) = e^{\frac{i}{\hbar} E t}.
\]  

(8)

The Laplacian \( \nabla^2 \) in polar \((r, \theta)\) and spherical \((r, \theta, \phi)\) coordinates is

\[
\nabla^2 = \begin{cases} \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}, & \text{for 2d}, \\ \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \left( \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right), & \text{for 3d}. \end{cases}
\]

(9)
In the following subsections the time-independent wave function is further separated into radial and angular wave functions. In particular, the radial wave equation can be written as associated Laguerre differential equation [9]

\[
\left( x \frac{d^2}{dx^2} + (v + 1 - x) \frac{d}{dx} + \lambda \right) L_\nu^m(x) = 0,
\]

where \( L_\nu^m(x) \) is the associated Laguerre polynomial [10]. For non-integer \( \nu \), \( L_\nu^m \) is also called generalized Laguerre function [11].

Two-dimensional quantum harmonic oscillator in polar coordinate

Substituting \( u(\vec{r}) = R(r)\Theta(\theta) \)

\[
\frac{1}{R(r)} \left[ \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} + \frac{1}{r^2} \left( \frac{1}{\Theta(\theta)} \frac{d^2}{d\theta^2} \right) \right] R(r) = E
\]

we can arrive at the equation for radial \( R(r) \) and angular wave function \( \Theta(\theta) \)

\[
\frac{1}{R(r)} \left( \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{l^2}{r^2} \right) R(r) = E,
\]

\[
\frac{1}{\Theta(\theta)} \frac{d^2}{d\theta^2} \Theta(\theta) = -l^2.
\]

The solution of Eq. (12b)

\[
\Theta(\theta) = \frac{1}{\sqrt{2\pi}} e^{i\theta}.
\]

Setting \( R(r) = r^l e^{-\frac{1}{2}a^2r^2} f(a^2r^2) \), substitution Eq. (12a) can be written in similar form to Eq. (10)

\[
u \frac{d^2}{d\nu^2} + (l + 1 - \nu) \frac{d}{d\nu} + \frac{E}{2\hbar \omega} - \frac{(l + 1)}{2} f(\nu),
\]

Therefore

\[
R(r) \propto r^l e^{-\frac{1}{2}a^2r^2} L_\nu^m(a^2r^2),
\]

and the quantization of eigen energy is similar to Eq. (5)

\[
E_\nu = \hbar \omega (2n + l + 1).
\]

Three-dimensional quantum harmonic oscillator in spherical coordinate

Substituting \( u(\vec{r}) = R(r)\Theta(\theta)\Phi(\phi) \)

\[
\frac{1}{R(r)} \left[ \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} + \frac{1}{r^2} \left( \frac{1}{\Theta(\theta)} \frac{d^2}{d\theta^2} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right) \right] R(r) = E
\]

we can arrive at the equation for radial \( R(r) \) and angular wave function \( \Theta(\theta) \)

\[
\frac{1}{R(r)} \left( \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} - \frac{\hbar l(l + 1)}{r^2} \right) R(r) = E,
\]

\[
\frac{1}{\Theta(\theta)} \left( \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) - \frac{m^2}{\sin^2 \theta} \right) \Theta(\theta) = -l(l + 1),
\]

\[
\frac{1}{\Phi(\phi)} \frac{d^2}{d\phi^2} \Phi(\phi) = -m^2.
\]

The solution of Eq. (17b) and (17c) is the spherical harmonics function [12], [13]

\[
\Theta(\theta)\Phi(\phi) = Y_{lm}(\theta, \phi).
\]

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Setting \( R(r) = r^l e^{-\frac{1}{2} \alpha^2 r^2} g(\alpha^2 r^2) \), substitution Eq. (12a) can be written in similar form to Eq. (10)

\[
\left( u \frac{d^2}{du^2} + (l + \frac{3}{2} - u) \frac{d}{du} + \frac{E}{2 \hbar \omega} - \frac{(l + 3/2)}{2} \right) g(u),
\]

Therefore

\[
R(r) \propto r^l e^{-\frac{1}{2} \alpha^2 r^2} L_n^{l+\frac{1}{2}}(\alpha^2 r^2),
\]

and the quantization of eigen energy is similar to Eq. (5)

\[
E_n = \hbar \omega \left( 2n + l + \frac{3}{2} \right).
\]

**RADIAL AND ANGULAR WAVE FUNCTIONS**

Eq. (51) and (21) show that the radial wave functions can be written in term of associated Laguerre polynomials. The normalization factors that satisfy

\[
1 = \begin{cases} 
\int_0^\infty [R(r)]^2 r dr, & \text{for } 2d, \\
\int_0^\infty [R(r)]^2 r^2 dr, & \text{for } 3d,
\end{cases}
\]

\[
\int_0^\infty x^k e^{-x} L_k^n(x) L_w^k(x) dx = \frac{\Gamma(v + k + 1)}{v!} \delta_{vw}
\]

Such that

\[
R(\vec{r}) = \begin{cases} 
\sqrt{\frac{n! 2(\alpha)^{2l+2}}{(n + l)!}} r^l e^{-\frac{1}{2} \alpha^2 r^2} L_n^l(\alpha^2 r^2), & \text{for } 2d, \\
\sqrt{\frac{n! 2(\alpha)^{2l+3}}{\Gamma(n + l + \frac{3}{2})}} r^l e^{-\frac{1}{2} \alpha^2 r^2} L_n^{l+\frac{1}{2}}(\alpha^2 r^2) & \text{for } 3d.
\end{cases}
\]

Here \( \Gamma(x) \) is the Gamma function.

Concerning the angular wave function Eqs. (12b) and (18b-c) can be associated with the eigen equations of the angular momentum in two and three dimensions, respectively [14]. Eq. (12b) is equivalent with the eigen equation

\[
\vec{L}_{2d} \theta(\theta) = \hbar \partial \theta(\theta)
\]

of the two-dimensional angular momentum

\[
\vec{L}_{2d} = -i \hbar \frac{\partial}{\partial \theta}.
\]

Meanwhile, Eq. (18b) is equivalent with the eigen equations

\[
\vec{L}_{3d} Y_{lm}(\theta, \phi) = \hbar^2 \{l(l + 1) Y_{lm}(\theta, \phi) + \hat{z} \cdot \vec{L}_{3d} Y_{lm}(\theta, \phi) = \hbar m Y_{lm}(\theta, \phi)
\]

of the three-dimensional angular momentum

\[
\vec{L}_{3d} = -i \hbar \left[ \hat{x} \left( - \sin \phi \frac{\partial}{\partial \theta} - \cot \theta \cos \phi \frac{\partial}{\partial \phi} \right) + \hat{y} \left( \cos \phi \frac{\partial}{\partial \theta} - \cot \theta \sin \phi \frac{\partial}{\partial \phi} \right) \right. \\
\left. + \hat{z} \frac{\partial}{\partial \phi} \right].
\]

Finally, the full time-space dependent eigen wave function is
\[ \psi = \begin{cases} 
\sqrt{n! (a^{2l+2} \pi (n + l)!} r^l e^{-\frac{1}{2}a^2r^2} L_n^l (a^2r^2)e^{i\theta} e^{-i\left(2n + l + 1\right)\omega t}, & \text{for } 2d, \\
\frac{n! 2(a^{2l+3})}{\Gamma\left(n + l + \frac{3}{2}\right)} r^l e^{-\frac{1}{2}a^2r^2} l_n^l \left(a^2r^2\right)Y_{lm}(\theta, \phi)e^{-i\left(2n + l + \frac{3}{2}\right)\omega t}, & \text{for } 3d. 
\end{cases} \] (28)

CONCLUSION

Two- and three-dimensional quantum oscillator harmonics is discussed in terms of radial and angular eigen wave functions. The angular wave functions are well-described using sinusoidal and spherical harmonics functions, respectively. Using appropriate ansatz, it is shown that the radial wave function is proportional to \( r^l e^{-\frac{1}{2}a^2r^2} L_n^l (a^2r^2) \), where \( l \) is related to the eigenvalues of angular momentum operators and \( L_n^l \) is generalized/associated Laguerre polynomials.

REFERENCES

