**Available online at:**
Al-Fiziya: Journal of Materials Science, Geophysics, Instrumentation and Theoretical Physics
Website: http://journal.uinjkt.ac.id/index.php/al-fiziya
Vol.5 No. II Tahun 2022, 95 - 100

# The radial wave function of 2D and 3D quantum harmonic oscillator

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Submitted: Agustus 2022; Revised: September 2022; Approved: Oktober 2022; Available Online: Desember 2022

**Abstract**. One dimensional quantum harmonic oscillator is well studied in elementary textbooks of quantum mechanics. The wave function of one-dimensional oscillator harmonic can be written in term of Hermite polynomial. Due to the symmetry of the spring energy, the wave functions of two-dimensional and three-dimensional harmonic oscillators can be written as products of the one-dimensional case. Because of that, the wave functions of two- and three-dimensional cases are focused on cartesian coordinates. In this article, we utilize polar and spherical coordinates to describe the wave function of two- and three-dimensional harmonic oscillators, respectively. The radial part of the wave functions can be written in term of associated Laguerre polynomials.

***Keywords***: *associated Laguerre polynomials, quantum harmonic oscillator, radial wave function, two dimension quantum harmonic oscillator, three dimension quantum harmonic oscillator.*

**DOI :** 10.15408/fiziya.v5i2.26172

## INTRODUCTION

In elementary textbooks of quantum mechanics, one dimensional quantum harmonic oscillator is well studied using a potential energy analogous to classical harmonic oscillator

$$V(x) = \frac{1}{2} M\omega^2 x^2. \tag{1}$$

Here $M$ is mass of the particle, $\omega$ is the characteristic radial frequency. The eigen wave function of the Schrödinger equation $(p^2/(2m) + V(x))\psi = i\hbar\dot{\psi}$ can be written in term of Hermite polynomials $H_n$ [1]

$$\psi_{1\mathrm{d}} = e^{-\frac{i}{\hbar}E_n t}\sqrt{\frac{\alpha}{2^n n!\sqrt{\pi}}}e^{-\frac{1}{2}\alpha^2 x^2}H_n(\alpha x). \tag{2}$$

Here $\alpha = \sqrt{M\omega/\hbar}$, and $E_n = \left(n + \frac{1}{2}\right)\hbar\omega$ is the quantized energy.

However, the potential is a central potential $V(\vec{r}) = \frac{1}{2}m\omega^2 r^2$ and therefore the eigen wave functions should be able to be written as a product of radial and angular wave functions [2]. The angular wave functions of two- and three-dimensional potentials wit central symmetry are well-described using sinusoidal and spherical harmonics

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P-ISSN: 2621-0215, E-ISSN: 2621-489X

functions, respectively [3]. However, studies of higher dimension quantum oscillators elude the discussion of radial wave function [4]–[8].

It may be due to the additive property of the norm of $\vec{r}$. Because of that, the generalization of the potential energy into two and three dimensions is straightforward

$$V(\vec{r}) = \begin{cases} \frac{1}{2}M\omega^2(x^2+y^2), & \text{for 2d,} \\ \frac{1}{2}M\omega^2(x^2+y^2+z^2), & \text{for 3d.} \end{cases} \tag{3}$$

Because of this symmetry, the wave functions of two- and three-dimensional cases are focused on cartesian coordinates

$$\begin{cases} \psi_{2\mathrm{d}} = e^{-i(n_x+n_y+1)\omega t}\left(\frac{\alpha}{2^n n!\sqrt{\pi}}\right)e^{-\frac{1}{2}\alpha^2(x^2+y^2)}H_{n_x}(\alpha x)H_{n_v}(\alpha y), \\ \psi_{3\mathrm{d}} = e^{-i\left(n_x+n_y+n_z+\frac{3}{2}\right)\omega t}\left(\frac{\alpha}{2^n n!\sqrt{\pi}}\right)^{\frac{3}{2}}e^{-\frac{1}{2}\alpha^2(x^2+y^2+z^2)}H_{n_x}(\alpha x)H_{n_v}(\alpha y)H_{n_z}(\alpha z). \end{cases} \tag{4}$$

Coefficient of time-dependent exponential indicates that the energy eigenvalues also have the additive properties.

$$E_n = \begin{cases} \hbar\omega\left(n_x+n_y+1\right), & \text{for 2d,} \\ \hbar\omega\left(n_x+n_y+n_z+\frac{3}{2}\right), & \text{for 3d.} \end{cases} \tag{5}$$

In this article, we utilize polar and spherical coordinates to describe the wave function of two- and three-dimensional harmonic oscillators, respectively. The understanding of radial wave function should provide better insight on the mathematical physics of basic quantum mechanics.

## VARIABLES SEPARATIONS OF SCHRÖDINGER EQUATION

Schrodinger equation of quantum particle under the influence of central potential energy

$$\left(-\frac{\hbar^2}{2M}\nabla^2+\frac{1}{2}m\omega^2r^2\right)\psi(\vec{r},t) = i\hbar\frac{\partial}{\partial t}\psi(\vec{r},t), \tag{6}$$

can be rewritten using variable separation with energy's eigen values $E$

$$\frac{1}{u(\vec{r})}\left(-\frac{\hbar^2}{2M}\nabla^2+\frac{1}{2}m\omega^2r^2\right)u(\vec{r}) = E = \frac{i\hbar}{T(t)}\frac{\partial T(t)}{\partial t}, \tag{7}$$

to arrive at the time-independent wave function $u(\vec{r})$ and space-independent $T(t)$. One can show that straight-forwardly shows that

$$T(t) = e^{-\frac{1}{\hbar}Et}. \tag{8}$$

The Laplacian $\nabla^2$ in polar $(r,\theta)$ and spherical $(r,\theta,\phi)$ coordinates is

$$\nabla^2 = \begin{cases} \frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}, & \text{for 2d,} \\ \frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right), & \text{for 3d.} \end{cases} \tag{9}$$

In the following subsections the time-independent wave function is further separated into radial and angular wave functions. In particular, the radial wave equation can be written as associated Laguerre differential equation [9]

$$\left(x\frac{d^2}{dx^2}+(\nu+1-x)\frac{d}{dx}+\lambda\right)L_\lambda^\nu(x), \tag{10}$$

where $L_\lambda^\nu(x)$ is the associated Laguerre polynomial [10]. For non-integer $\nu$, $L_\lambda^\nu$ is also called generalized Laguerre function [11].

## Two-dimensional quantum harmonic oscillator in polar coordinate

Substituting $u(\vec{r})=R(r)\Theta(\theta)$

$$\frac{1}{R(r)}\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}+\frac{1}{r^2}\left[\frac{1}{\Theta(\theta)}\frac{d^2\Theta(\theta)}{d\theta^2}\right]\right)R(r)=E \tag{11}$$

we can arrive at the equation for radial $R(r)$ and angular wave function $\Theta(\theta)$

$$\frac{1}{R(r)}\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{l^2}{r^2}\right)R(r)=E, \tag{12a}$$

$$\frac{1}{\Theta(\theta)}\frac{d^2\Theta(\theta)}{d\theta^2}=-l^2. \tag{12b}$$

The solution of Eq. (12b)

$$\Theta(\theta)=\frac{1}{\sqrt{2\pi}}e^{il\theta}. \tag{13}$$

Setting $R(r)=r^l e^{-\frac{1}{2}\alpha^2r^2}f(\alpha^2r^2)$, substitution Eq. (12a) can be written in similar form to Eq. (10)

$$\left(u\frac{d^2}{du^2}+(l+1-u)\frac{d}{du}+\frac{E}{2\hbar\omega}-\frac{(l+1)}{2}\right)f(u), \tag{14}$$

Therefore

$$R(r)\propto r^l e^{-\frac{1}{2}\alpha^2r^2}L_n^l(\alpha^2r^2), \tag{15}$$

and the quantization of eigen energy is similar to Eq. (5)

$$E_\nu=\hbar\omega(2n+l+1). \tag{16}$$

## Three-dimensional quantum harmonic oscillator in spherical coordinate

Substituting $u(\vec{r})=R(r)\Theta(\theta)\Phi(\phi)$

$$\frac{1}{R(r)}\left(\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}+\frac{1}{r^2}\left[\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]\right)R(r)=E \tag{17}$$

we can arrive at the equation for radial $R(r)$ and angular wave function $\Theta(\theta)$

$$\frac{1}{R(r)}\left(\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}-\frac{\hbar l(l+1)}{r^2}\right)R(r)=E, \tag{18a}$$

$$\frac{1}{\Theta(\theta)}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)-\frac{m^2}{\sin^2\theta}\right)\Theta(\theta)=-l(l+1), \tag{18b}$$

$$\frac{1}{\Phi(\phi)}\frac{d^2\Phi(\phi)}{d\phi^2}=-m^2. \tag{18c}$$

The solution of Eq. (17b) and (17c) is the spherical harmonics function [12], [13]

$$\Theta(\theta)\Phi(\phi)=Y_{lm}(\theta,\phi). \tag{19}$$

Setting $R(r) = r^l e^{-\frac{1}{2}\alpha^2 r^2} g(\alpha^2 r^2)$, substitution Eq. (12a) can be written in similar form to Eq. (10)

$$\left(u\frac{d^2}{du^2} + \left(l + \frac{3}{2} - u\right)\frac{d}{du} + \frac{E}{2\hbar\omega} - \frac{(l+3/2)}{2}\right) g(u), \tag{20}$$

Therefore

$$R(r) \propto r^l e^{-\frac{1}{2}\alpha^2 r^2} L_n^{l+\frac{1}{2}}(\alpha^2 r^2), \tag{21}$$

and the quantization of eigen energy is similar to Eq. (5)

$$E_\nu = \hbar\omega\left(2n + l + \frac{3}{2}\right). \tag{22}$$

## RADIAL AND ANGULAR WAVE FUNCTIONS

Eq. (51) and (21) show that the radial wave functions can be written in term of associated Laguerre polynomials. The normalization factors that satisfy

$$1 = \begin{cases} \int_0^\infty [R(r)]^2\, r dr, & \text{for 2d,} \\ \int_0^\infty [R(r)]^2\, r^2 dr, & \text{for 3d,} \end{cases} \tag{23}$$

can be found using the orthogonality of associated Laguerre polynomials

$$\int_0^\infty x^k e^{-x} L_\nu^k(x) L_w^k(x)\, dx = \frac{\Gamma(\nu + k + 1)}{\nu!}\delta_{\nu w} \tag{24}$$

Such that

$$R(\vec{r}) = \begin{cases} \sqrt{\frac{n!\,2(a)^{2l+2}}{(n+l)!}}\, r^l e^{-\frac{1}{2}\alpha^2 r^2} L_n^l(\alpha^2 r^2), & \text{for 2d,} \\ \sqrt{\frac{n!\,2(a)^{2l+3}}{\Gamma\left(n + l + \frac{3}{2}\right)}}\, r^l e^{-\frac{1}{2}\alpha^2 r^2} L_n^{l+\frac{1}{2}}(\alpha^2 r^2) & \text{for 3d.} \end{cases} \tag{23}$$

Here $\Gamma(x)$ is the Gamma function.

Concerning the angular wave function Eqs. (12b) and (18b-c) can be associated with the eigen equations of the angular momentum in two and three dimensions, respectively [14]. Eq. (12b) is equivalent with the eigen equation

$$\vec{L}_{2\mathrm{d}}\Theta(\theta) = \hbar l \Theta(\theta) \tag{24}$$

of the two-dimensional angular momentum

$$\vec{L}_{2\mathrm{d}} = -i\hbar\frac{\partial}{\partial\theta}. \tag{25}$$

Meanwhile, Eq. (18b) is equivalent with the eigen equations

$$L_{3\mathrm{d}}^2 \mathrm{Y}_{lm}(\theta,\phi) = \hbar^2 l(l+1)\mathrm{Y}_{lm}(\theta,\phi) \tag{26}$$
$$\hat{z}\cdot\vec{L}_{3d}\mathrm{Y}_{lm}(\theta,\phi) = \hbar m \mathrm{Y}_{lm}(\theta,\phi)$$

of the three-dimensional angular momentum

$$\vec{L}_{3\mathrm{d}} = -i\hbar\left[\hat{x}\left(-\sin\phi\frac{\partial}{\partial\theta} - \cot\theta\cos\phi\frac{\partial}{\partial\phi}\right) + \hat{y}\left(\cos\phi\frac{\partial}{\partial\theta} - \cot\theta\sin\phi\frac{\partial}{\partial\phi}\right) + \hat{z}\frac{\partial}{\partial\phi}\right]. \tag{27}$$

Finally, the full time-space dependent eigen wave function is

$$\psi = \begin{cases} \sqrt{\frac{n!\,(a)^{2l+2}}{\pi(n+l)!}}\, r^l e^{-\frac{1}{2}\alpha^2 r^2} L_n^l(\alpha^2 r^2) e^{il\theta} e^{-i(2n+l+1)\omega t}, & \text{for 2d,} \\ \sqrt{\frac{n!\,2(a)^{2l+3}}{\Gamma\left(n+l+\frac{3}{2}\right)}}\, r^l e^{-\frac{1}{2}\alpha^2 r^2} L_\nu^{l+\frac{1}{2}}(\alpha^2 r^2) Y_{lm}(\theta,\phi) e^{-i\left(2n+l+\frac{3}{2}\right)\omega t}, & \text{for 3d.} \end{cases} \quad (28)$$

## CONCLUSION

Two- and three-dimensional quantum oscillator harmonics is discussed in terms of radial and angular eigen wave functions. The angular wave functions are well-described using sinusoidal and spherical harmonics functions, respectively. Using appropriate ansatz, it is shown that the radial wave function is proportional to $r^l e^{-\frac{1}{2}\alpha^2 r^2} L_n^k(\alpha^2 r^2)$, where $l$ is related to the eigenvalues of angular momentum operators and $L_n^k$ is generalized/associated Laguerre polynomials.

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