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Improved Steepest Descent Method using Modified Bessel Function $K_{1/4}$ for Gamma Function Evaluation

Adam B. Cahaya [1,†]

[1]Department of Physics, Faculty of Mathematics and Natural Sciences, Universitas Indonesia
Depok 16424, Indonesia

[†]adam@sci.ui.ac .id

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Abstrak. Metode *Steepest descent* (turunan tercuram menggunakan aproksimasi fungsi Gaussian $Ae^{-B(x-C)^2}$ketika memperkirakan nilai integral dari sebuah fungsi. Dalam artickel ini kami meningkatkan aproksimasi ini dengan menggunakan fungsi berbentuk $Ae^{-B(x-C)^2-D(x-C)^4}$. Sebagai contoh, kami melakukan aproksimasi terhadap nilai fungsi gamma untuk mendapatkan aproksimasi yang lebih baik daripada Stirling formula yang sering digunakan untuk estimasi nilai faktorial dari bilangan besar.

Kata Kunci: *metode Steepest descent, formula Stirling, fungsi gamma..*

Abstract. Steepest descent method employs a Gaussian function $Ae^{-B(x-C)^2}$ when approximating an integral of a function. In this article we improve the approximation by using function in the form of $Ae^{-B(x-C)^2-D(x-C)^4}$. As an example, we approximate the value of gamma function to provide improved approximation for Stirling formula that is often used for estimating factorial of a large number.

Keywords: *Steepest descent method, Stirling formula, gamma function.*

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INTRODUCTION

Factorial of a positive integer is defined as the following product.

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1 \tag{1}$$

This definition can be extended to complex and real number by using gamma function.

$$z! = \Gamma(z+1) = \int_0^\infty t^z e^{-t} dt \tag{2}$$

The evaluation of the factorial of a large number by calculating the product of positive integers require a very long time. The most used formula for approximation of the large factorials is the following Stirling's formula

$$n! \approx \sqrt{2\pi} n^{n+\frac{1}{2}} e^{-n} \tag{3}$$

Slightly better approximation formulas have also been developed [1].

$$n! \approx \sqrt{2\pi} (n+p)^{n+\frac{1}{2}} e^{-(n+p)}, \tag{4}$$

Where p is constant. In particular, $p = 0$ corresponds to Stirling formula, while $p = \frac{1}{2}$ and $\frac{1}{2} \pm \frac{1}{\sqrt{12}}$ are also known as Burnside's formula [2] and Schuster formula [3], respectively. Eq. (4) can be derived by approximating $t^z e^{-t}$ inside the integral expression of gamma function using steepest descent method [4] , as illustrated in Fig. 1.

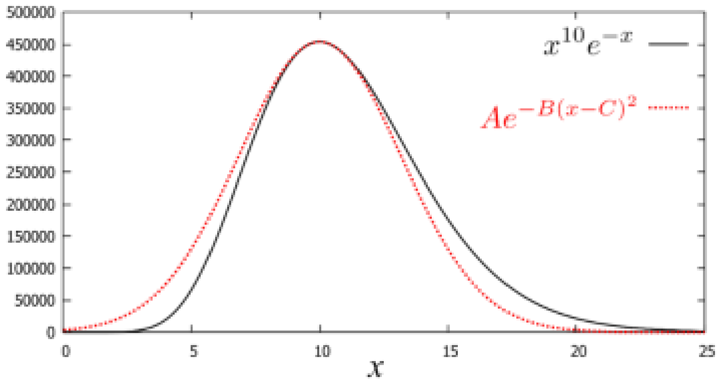

Figure 1. Approximation of $10! = \int_0^\infty x^{10} e^{-x}\, dx$ using $\int_{-\infty}^\infty Ae^{-B(x-C)^2}\, dx$

In the steepest descent method, $t^z e^{-t}$ function is approximated using Gaussian function in the form of $Ae^{-B(x-C)^2}$. While the steepest descent is one of the simplest and best known methods for approximating a function[5], [6], how to improve this method is still an ongoing research question[7], [8]. In this article we show that the Steepest descent method can be improved by using higher order exponential function $Ae^{-B(x-C)^2 - D(x-C)^2}$. In particular, we demonstrate that a better approximation than Stirling formula can be obtained by approximating the expression for gamma function for a large number with $Ae^{-B(x-C)^2 - D(x-C)^2}$.

METHODS

Approximation using $Ae^{-B(x-C)^2}$ and $Ae^{-B(x-C)^2 - D(x-C)^2}$

Let us first discuss the steepest descent approximation for generalized Stirling formula (Eq. 4). By using variable substitution $t = x^m$, gamma function in Eq. (2) can be written in the following integral

$$n! = m\int_0^\infty x^{mn+m-1}e^{-x^m}\,dx = \int_0^\infty e^{\log m+(mn+m-1)\log x - x^m}\,dx. \tag{5}$$

In the steepest descent, this integral can be evaluated by taking the Taylor expansion of the function inside the exponential

$$\log m + (mn+m-1)\log x - x^m = \log A - B(x-C)^2 + O((x-C)^3), \tag{6}$$

where

$$A = m\left(n+1-\frac{1}{m}\right)^{\left(n+1-\frac{1}{m}\right)} e^{-\left(n+1-\frac{1}{m}\right)}, \tag{7a}$$

$$B = \frac{m^2}{2}\left(n+1-\frac{1}{m}\right)^{1-\frac{2}{m}}, \tag{7b}$$

$$C = \left(n+1-\frac{1}{m}\right)^{\frac{1}{m}}. \tag{7c}$$

By ignoring $O((x-C)^3)$ we can approximate the integral using the following equation.

$$\int_{-C}^\infty e^{-Bx^2}dx \approx \int_{-\infty}^\infty e^{-Bx^2}\,dx = \sqrt{2\pi}\left(n+1-\frac{1}{m}\right)^{n+\frac{1}{2}} e^{-\left(n+1-\frac{1}{m}\right)} \tag{8}$$

One can see that Eqs. (3) and (4) are obtained when $m = 1$ and $(1-p)^{-1}$, respectively.

This approximation can be improved by choosing $m = 3$. In this case the expansion coefficient of $(x-C)^3$ is zero and we can approximate the function into higher order, as illustrated in Fig. 2.

$$\log 3 + (3n+2)\log x - x^3 = \log A - B(x-C)^2 - D(x-C)^4 + O((x-C)^5) \tag{9}$$

where

$$A_{m=3} = 3\left(n+\frac{2}{3}\right)^{\left(n+\frac{2}{3}\right)} e^{-\left(n+\frac{2}{3}\right)} \tag{10a}$$

$$B_{m=3} = \frac{9}{2}\left(n+\frac{2}{3}\right)^{\frac{1}{3}} \tag{10b}$$

$$C_{m=3} = \left(n+\frac{2}{3}\right)^{\frac{1}{3}} \tag{10c}$$

$$D = \frac{3}{4}\left(n+\frac{2}{3}\right)^{-\frac{1}{3}} \tag{10d}$$

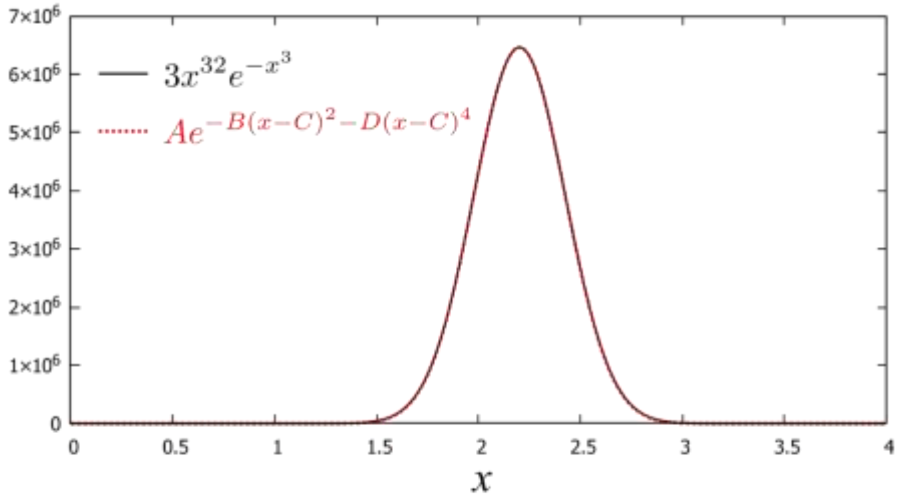

Figure 2. Approximation of $10! = 3\int_0^\infty x^{32}e^{-x^3}\,dx$ using $\int_{-\infty}^\infty Ae^{-B(x-C)^2-D(x-C)^4}\,dx$

In this case, we can approximate the integral using the following equation.

$$\int_{-C}^{\infty} e^{-Bx^2-Dx^4}dx \approx \int_{-\infty}^{\infty} e^{-Bx^2-Dx^4}dx = \frac{1}{2}\sqrt{\frac{B}{D}}e^{\frac{B^2}{8D}}K_{\frac{1}{4}}\left(\frac{B^2}{8D}\right) \tag{11}$$

Here $K_\alpha(x)$ is the modified Bessel function of the second kind, which is defined as follows

$$K_\alpha(x) = \int_0^{\infty} e^{-x\cosh t}\cosh \alpha t\, dt \tag{12}$$

The asymptotic form of $K_\alpha(x)$ for large x is[9], [10]

$$K_\alpha(x) = \sqrt{\frac{\pi}{2x}}e^{-x}\left(1+\frac{4\alpha^2-1}{8x}+\frac{(4\alpha^2-1)(4\alpha^2-3^2)}{2!\,(8x)^2} + \frac{(4\alpha^2-1)(4\alpha^2-3^2)(4\alpha^2-5^2)}{3!\,(8x)^3}+\cdots\right) \tag{13}$$

Therefore

$$\int_{-\infty}^{\infty} e^{-Bx^2-Dx^4}dx = \sqrt{\frac{\pi}{B}}\left(1+\frac{\frac{1}{4}-1}{B^2}D+\frac{\left(\frac{1}{4}-1\right)\left(\frac{1}{4}-3^2\right)}{2!\,B^4}D^2 + \frac{\left(\frac{1}{4}-1\right)\left(\frac{1}{4}-3^2\right)\left(\frac{1}{4}-5^2\right)}{3!\,B^6}D^3+\cdots\right) \tag{14}$$

RESULT AND DISCUSSION

We can improve the generalized Stirling formula by using $Ae^{-B(x-C)^2-D(x-C)^4}$ to approximate function inside the integral expression for gamma function. By substituting Eqs. (10a)-(10d), we can arrive at the following improved approximation.

$$n! \approx 3\left(n+\frac{2}{3}\right)^{n+1} e^{-\left(n+\frac{2}{3}\right)}\frac{\sqrt{6}}{2}e^{\frac{27}{8}\left(n+\frac{2}{3}\right)}K_{\frac{1}{4}}\left(\frac{27}{8}\left(n+\frac{2}{3}\right)\right) \tag{15}$$

Furthermore, we can use the asymptotic form of $K_\alpha(x)$ to estimate the corrections

$$= \left(n+\frac{2}{3}\right)^{n+\frac{1}{2}} e^{-\left(n+\frac{2}{3}\right)}\sqrt{2\pi}\left(1+\frac{\frac{1}{4}-1}{27\left(n+\frac{2}{3}\right)}+\frac{\left(\frac{1}{4}-1\right)\left(\frac{1}{4}-3^2\right)}{2!\,27^2\left(n+\frac{2}{3}\right)^2} + \frac{\left(\frac{1}{4}-1\right)\left(\frac{1}{4}-3^2\right)\left(\frac{1}{4}-5^2\right)}{3!\,27^3\left(n+\frac{2}{3}\right)^3}+\cdots\right) \tag{16}$$

The accuracy of this approximation is illustrated in the Table 1.

Table 1 Accuracy of approximation

n	$n!$	$\sqrt{2\pi}n^{n+\frac{1}{2}}e^{-n}$	$\left(n+\frac{2}{3}\right)^{n+\frac{1}{2}} e^{-\left(n+\frac{2}{3}\right)}\sqrt{2\pi}\left(1+\frac{\frac{1}{4}-1}{27\left(n+\frac{2}{3}\right)}+\cdots\right)$	**Deviation (%)**

1	1	0.92	1.00	0.202%
2	2	1.92	2.00	0.199%
4	24	23.51	23.96	0.148%
8	40320	39902.40	40283.02	0.092%
10	3.629×10^{6}	3.599×10^{6}	3.626×10^{6}	0.077%
20	2.433×10^{18}	2.423×10^{18}	2.432×10^{18}	0.042%
40	8.159×10^{47}	8.142×10^{47}	8.157×10^{47}	0.022%
80	7.157×10^{118}	7.149×10^{118}	7.156×10^{118}	0.011%
100	9.333×10^{157}	9.325×10^{157}	9.332×10^{157}	0.009%

Here we note that $O((x-C)^5)$ terms in Eq. (18) can be included for better approximation by expansion of $e^{O((x-C)^5)}$ and utilize the following relations.

$$\int_{-\infty}^{\infty} e^{-Bx^2-Dx^4}\, x^{4s} dx = (-1)^s \sqrt{\frac{\pi}{B}} \frac{\partial^s}{\partial D^s}\left(1+\frac{\frac{1}{4}-1}{B^2}D+\frac{\left(\frac{1}{4}-1\right)\left(\frac{1}{4}-3^2\right)}{2!\,B^4}D^2 + \frac{\left(\frac{1}{4}-1\right)\left(\frac{1}{4}-3^2\right)\left(\frac{1}{4}-5^2\right)}{3!\,B^6}D^3+\cdots\right) \tag{16a}$$

$$\int_{-\infty}^{\infty} e^{-Bx^2-Dx^4}\, x^{4s+2} dx = (-1)^{s+1} \frac{\partial}{\partial B}\sqrt{\frac{\pi}{B}} \frac{\partial^s}{\partial D^s}\left(1+\frac{\frac{1}{4}-1}{B^2}D+\frac{\left(\frac{1}{4}-1\right)\left(\frac{1}{4}-3^2\right)}{2!\,B^4}D^2 + \frac{\left(\frac{1}{4}-1\right)\left(\frac{1}{4}-3^2\right)\left(\frac{1}{4}-5^2\right)}{3!\,B^6}D^3+\cdots\right) \tag{16b}$$

CONCLUSION

To summarize, we review the derivation of Stirling formula by using steepest descent method for approximating gamma function. We can arrive at the Stirling approximation and other and related formulas by substituting the appropriate m into Eq. (8).

Table 2. Values of m for Stirling-like approximations

m	**Formula**
1	Stirling formula $\sqrt{2\pi}n^{n+\frac{1}{2}}e^{-n}$
2	Burnside formula $\sqrt{2\pi}\left(n+\frac{1}{2}\right)^{n+\frac{1}{2}}e^{-\left(n+\frac{1}{2}\right)}$

$\frac{2\sqrt{3}}{\sqrt{3}\mp 1}$	Schuster formula $\sqrt{2\pi}e^{-\left(n+\frac{1}{2}\pm\frac{1}{\sqrt{12}}\right)}\left(n+\frac{1}{2}\pm\frac{1}{\sqrt{12}}\right)^{n+\frac{1}{2}}$
$\frac{1}{1-p}$	$\sqrt{2\pi}(n+p)^{n+\frac{1}{2}}e^{-(n+p)}$

We show that we can improve the steepest descent approximation by replacing Gaussian function $Ae^{-B(x-C)^2}$ with higher exponential term $Ae^{-B(x-C)^2-D(x-C)^4}$. By applying the improved method to integral expression of gamma function, we can arrive at the improved Stirling approximation that utilize modified Bessel function of the second kind $K_{1/4}$ (see Eq. 15). The accuracy of this approximation can be seen in the Table 1.

REFERENCES

[1] W. A. Bowers, "More about approximations to n!," *Am. J. Phys.*, vol. 51, no. 9, p. 778, Sep. 1983, doi: 10.1119/1.13141.

[2] W. Burnside, "A rapidly convergent series for log N," *Messenger Math*, vol. 46, no. 1, pp. 157–159, 1917.

[3] W. Schuster, "Improving Stirling's formula," *Arch. der Math.*, vol. 77, pp. 170–176, 2001.

[4] C. Leubner, "Generalised Stirling approximations to N!," *Eur. J. Phys.*, vol. 6, no. 4, pp. 299–301, Oct. 1985, doi: 10.1088/0143-0807/6/4/016.

[5] S. S. Petrova and A. D. Solov'ev, "The Origin of the Method of Steepest Descent," *Hist. Math.*, vol. 24, no. 4, pp. 361–375, 1997, doi: https://doi.org/10.1006/hmat.1996.2146.

[6] J. C. Meza, "Steepest descent," *WIREs Comput. Stat.*, vol. 2, no. 6, pp. 719–722, 2010, doi: https://doi.org/10.1002/wics.117.

[7] Y. Pu and J. Wang, "Fractional-order global optimal backpropagation machine trained by an improved fractional-order steepest descent method," *Front. Inf. Technol. Electron. Eng.*, vol. 21, no. 6, pp. 809–833, 2020, doi: 10.1631/FITEE.1900593.

[8] J.-C. Son, M.-K. Baek, S.-H. Park, and D.-K. Lim, "Improved Immune Algorithm Combined with Steepest Descent Method for Optimal Design of IPMSM for FCEV Traction Motor," *Energies*, vol. 14, no. 13, 2021, doi: 10.3390/en14133904.

[9] E. J. Weniger and J. Čížek, "Rational approximations for the modified Bessel function of the second kind," *Comput. Phys. Commun.*, vol. 59, no. 3, pp. 471–493, 1990, doi: https://doi.org/10.1016/0010-4655(90)90089-J.

[10] Z.-H. Yang and Y.-M. Chu, "On approximating the modified Bessel function of the second kind," *J. Inequalities Appl.*, vol. 2017, no. 1, p. 41, 2017, doi: 10.1186/s13660-017-1317-z.