Ohmic and Diamagnetic Currents Contribution on the Electromagnetic Penetration Depth of a Conducting Surface

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Abstrak. Karena memiliki elektron konduksi, logam merupakan reflektor yang baik untuk gelombang elektromagnetik. Gelombang elektromagnetik yang menembus permukaan logam memiliki kedalaman tembus yang terbatas. Ada dua limit yang sering dipelajari dengan baik dalam buku teks fisika. Limit pertama adalah limit gelombang elektromagnetik frekuensi tinggi yang menembus logam dengan konduktivitas kecil dan medan statis (frekuensi rendah). Limit kedua adalah gelombang elektromagnetik yang menembus superkonduktor (logam dengan konduktivitas sangat besar). Dalam artikel ini kami mempelajari daerah pertengahan antara dua limit ekstrim ini. Dengan menetapkan kerapatan arus listrik sebagai jumlah total arus Ohmik dan diamagnetik, kami menunjukkan transisi antara dua batas ini.

Kata Kunci: Penetrasi Elektromagnetik, skin depth, diamagnetic current, Maxwell equations, London equation.

Abstract. Due to its conducting electron, metal is a good reflector for electromagnetic wave. An electromagnetic wave penetrating a metallic surface has a finite penetrating depth. There are two limit that are well studied in the physics textbooks. They are high frequency electromagnetic wave penetrating a metal with small conductivity and a static (low frequency) field penetrating a superconductor (metal with infinitely large conductivity). In this article we study the intermediate regime between these two limits. By setting the electric current density as the total sum of both Ohmic and Diamagnetic currents, we derive the penetration depth in the intermediate regime., we show the transition between these two limits.

Keywords: Electromagnetic penetration, skin depth, diamagnetic current, Maxwell equations, London equation.

INTRODUCTION

Due to its conducting nature, electromagnetic waves are reflected by metallic surface. The reflection mechanism can be mathematically described by substituting the following Ohmic current into the Maxwell equation[1].

$$\mathbf{J} = \sigma\mathbf{E}, \tag{1}$$

where $\mathbf{J}$ is electric current density, σ is conductivity and $\mathbf{E}$ is the electric field. In the limit of small conductivity, the skin depth of the penetrating electromagnetic wave is inversely proportional to the conductivity of the metal[2].

$$\delta = \frac{2}{\sigma}\sqrt{\frac{\varepsilon}{\mu}}. \tag{2}$$

Here ε is the permittivity and μ is the permeability. Because of that, one can expect that the skin depth of superconductor is zero. However, Meisner effect shows that a penetrating magnetic field has a finite skin depth, also known as London penetration depth l, is not zero [3], [4].

$$l = \sqrt{\frac{m}{\mu n e^2}}. \tag{3}$$

Here m is the electron mass, $-e$ is the electron charge and n is the carrier density. London penetration can be derived by substituting the following London diamagnetic current into the Maxwell equation and taking a static limit [5]–[7]. The reflection of electromagnetic field in superconductor is mainly caused by the London diamagnetic current

$$\mathbf{J} = -\frac{n e^2}{m}\mathbf{A}. \tag{4}$$

Here $\mathbf{A}$ is the vector potential.

δ and λ are two limiting cases that are well studied in the physics textbooks [1], [8]. In this article, we focus on the intermediate regime where the effect of both ohmic and diamagnetic current are important. The effect of Ohmic current has been widely studied in surface designs [9], [10]. On the other hand, the study of diamagnetic current is gaining increasing attention in physics [11], [12] and related areas [13], [14].

METHOD: MATHEMATICAL FORMALISM

The dynamics of electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$ in a material is best described by Maxwell equations. The Maxwell equations can be written in term of scalar potential ϕ and vector potential $\mathbf{A}$.

$$\mathbf{E} = -\boldsymbol{\nabla}\phi - \frac{\partial \mathbf{A}}{\partial t}, \tag{5a}$$

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}, \tag{5b}$$

By using Lorentz gauge condition,

$$\boldsymbol{\nabla} \cdot \mathbf{A} + \varepsilon\mu\frac{\partial \phi}{\partial t} = \mathbf{0}, \tag{6}$$

one can obtain the partial differential equations for ϕ and vector potential $\mathbf{A}$ [15]

$$\left(\nabla^2 - \frac{\partial^2}{c^2 \partial t^2}\right)\phi = -\frac{\rho}{\varepsilon}, \tag{6a}$$

$$\left(\nabla^2 - \frac{\partial^2}{c^2 \partial t^2}\right)\mathbf{A} = -\mu\mathbf{J}. \tag{6b}$$

Here $c = 1/\sqrt{\varepsilon\mu}$ is the speed of light in the material. We will solve Eqs. (6a) and (b) by choosing $\phi = 0$. In this case one can show that for electromagnetic wave propagating in x direction, the solution of $\mathbf{A}, \mathbf{E}$ and $\mathbf{B}$ that have the following form

$$\mathbf{A} = A\hat{\mathbf{y}}e^{i(kx-\omega t)} \tag{7a}$$

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} = i\omega A\hat{\mathbf{y}}e^{i(kx-\omega t)} \tag{7b}$$

$$\mathbf{B} = \nabla \times \mathbf{A} = ikA\hat{\mathbf{z}}e^{i(kx-\omega t)} \tag{7c}$$

Here k is the wavenumber and ω is the frequency. Using this choice of ϕ and vector potential $\mathbf{A}$, we can derive the penetration depth by examining the imaginary part of k because it creates an exponentially decaying term in the electromagnetic fields. The inverse of this imaginary term is the penetration depth.

In the following subsections we illustrate the derivation of these two limits: high frequency electromagnetic wave penetrating a metal with small conductivity and a low frequency field penetrating a metal with large conductivity, by consider Ohmic and diamagnetic current, respectively. In the next Section we will combine both currents to see the penetration depth for intermediate regimes.

Reflection by Ohmic current

To study the limit of high frequency electromagnetic wave penetrating a metal with small conductivity, we solve Eq. (6b) by substiting Eq. (1).

$$\left(\nabla^2 - \frac{\partial^2}{c^2 \partial t^2}\right)\mathbf{A} = -\mu\mathbf{J}_{\text{Ohm}} = \mu\sigma\frac{\partial \mathbf{A}}{\partial t} \tag{8}$$

Substituting Eq. (7) to Eq. (8) we arrive at the following equation

$$\left(-k^2 + \frac{\omega^2}{c^2} + i\mu\sigma\omega\right)A\hat{\mathbf{y}}e^{i(kx-\omega t)} = 0 \tag{9}$$

The dispersion $k(\omega)$ can then be derived as follows.

$$k = \frac{\omega}{c}\sqrt{1 + \frac{i\sigma}{\varepsilon\omega}} \tag{10}$$

For small conductivity and large frequency $\sigma \ll \varepsilon\omega$, one can show that the imaginary part of k is proportional to σ

$$\lim_{\sigma \ll \varepsilon\omega} k = \frac{\omega}{c}\left(1 + \frac{i\sigma}{2\varepsilon\omega}\right) = \frac{\omega}{c} + i\frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}} \tag{11}$$

The inverse of this imaginary term is the penetration depth in Eq. (2).

Reflection by Diamagnetic current

To study the limit of low frequency electromagnetic wave penetrating a metal with large conductivity, we solve Eq. (6b) by substiting Eq. (4).

$$\left(\nabla^2 - \frac{\partial^2}{c^2 \partial t^2}\right)\mathbf{A} = -\mu\mathbf{J}_{\text{Dia}} = \mu\frac{ne^2}{m}\mathbf{A} \tag{12}$$

Substituting Eq. (7a) to Eq. (12) we arrive at the following equation

$$\left(-k^2 + \frac{\omega^2}{c^2} - \mu\frac{ne^2}{m}\right)A\hat{\mathbf{y}}e^{i(kx-\omega t)} = 0 \tag{13}$$

The dispersion $k(\omega)$ can then be derived as follows.

$$k = \frac{\omega^2}{c^2} - \mu\frac{ne^2}{m} \tag{15}$$

For small frequency, k is purely imaginary

$$\lim_{\omega \to 0} k = i\mu\frac{ne^2}{m} \tag{16}$$

Its inverse is the London penetration depth in Eq. (3).

RESULT AND DISCUSSION: REFLECTION BY OHMIC AND DIAMAGNETIC CURRENT

To study the intermediate regime, we set the electric current density as the total sum of both Ohmic and Diamagnetic currents.

$$\mathbf{J} = \mathbf{J}_{\text{Ohm}} + \mathbf{J}_{\text{Dia}} = \sigma\mathbf{E} - \frac{ne^2}{m}\mathbf{A} \tag{17}$$

In terms of δ and l:

$$\mu\mathbf{J} = \frac{2}{c\delta}\frac{\partial\mathbf{A}}{\partial t} - \frac{1}{l^2}\mathbf{A} \tag{18}$$

We can solve the expression for $\mathbf{A}$ by substituting Eq. (8) back to Eq. (6b)

$$\left(\nabla^2 - \frac{\partial^2}{c^2\partial t^2}\right)\mathbf{A} = -\frac{2}{c\delta}\frac{\partial\mathbf{A}}{\partial t} + \frac{1}{l^2}\mathbf{A} \tag{19}$$

Substituting Eq. (7a) to Eq. (12) we arrive at the following equation

$$\left(-k^2 + \frac{\omega^2}{c^2} + i\frac{2\omega}{c\delta} - \frac{1}{l^2}\right)A\hat{\mathbf{y}}e^{i(kx-\omega t)} = 0 \tag{20}$$

The dispersion $k(\omega)$ can then be derived as follows.

$$k = \sqrt{\frac{\omega^2}{c^2} - \frac{1}{l^2} + i\frac{2\omega}{c\delta}} \tag{21}$$

To discuss the penetration depth *i.e.* the inverse of the imaginary part of k, we need to look at two frequency regimes separated by the critical frequency $\omega_0 = c/l$

High frequency $\omega > c/l$

In this case, the dispersion $k(\omega)$ can be written as follows

$$k = \frac{1}{c}\left((\omega^2 - \omega_0^2)^2 + \left(\frac{2c\omega}{\delta}\right)^2\right)^{\frac{1}{4}}e^{i\alpha/2} \tag{22a}$$

$$\alpha = \tan^{-1}\frac{2c\omega}{(\omega^2 - \omega_0^2)\delta} \tag{22b}$$

Its imaginary part is as follows.

$$\text{Im}\,k = \frac{1}{c}\left((\omega^2 - \omega_0^2)^2 + \left(\frac{2c\omega}{\delta}\right)^2\right)^{\frac{1}{4}}\sin\frac{\alpha}{2} \tag{23a}$$

$$= \frac{1}{c\sqrt{2}}\sqrt{\sqrt{(\omega^2 - \omega_0^2)^2 + \left(\frac{2c\omega}{\delta}\right)^2} - (\omega^2 - \omega_0^2)} \tag{23b}$$

Low frequency $\omega < c/l$

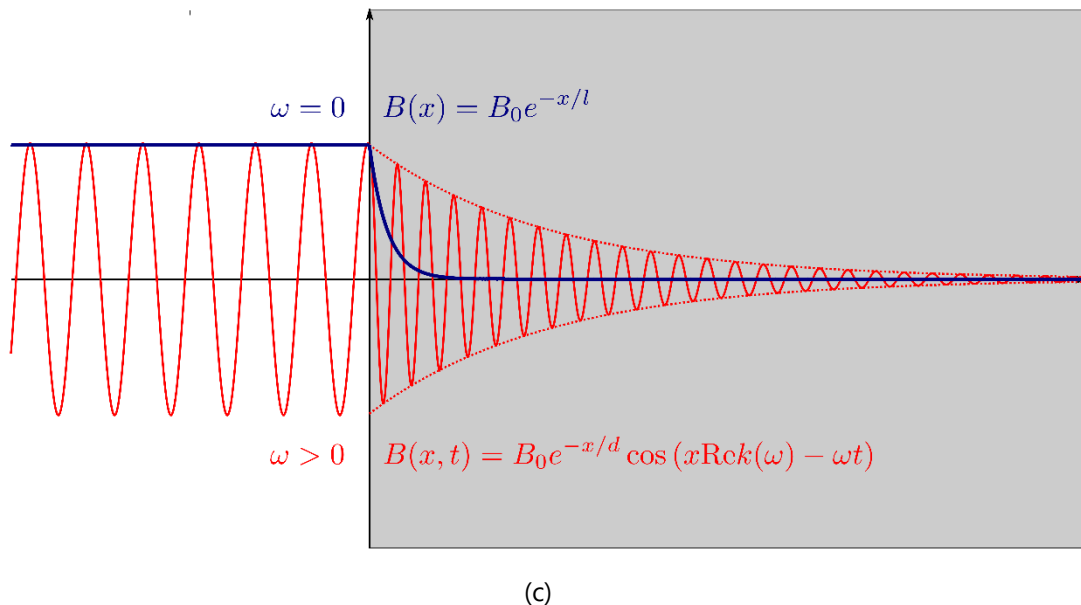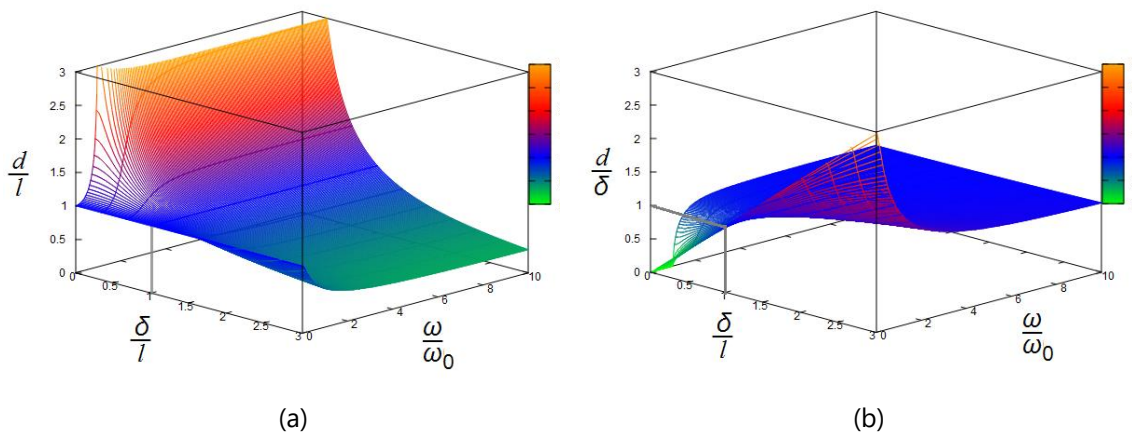In this case, the dispersion $k(\omega)$ can be written as follows

$$k = \frac{i}{c}\left((\omega_0^2 - \omega^2)^2 + \left(\frac{2c\omega}{\delta}\right)^2\right)^{\frac{1}{4}} e^{-i\beta/2} \tag{24a}$$

$$\beta = \tan^{-1}\frac{2c\omega}{(\omega_0^2 - \omega^2)\delta} \tag{24b}$$

Its imaginary part is as follows.

$$\text{Im } k = \frac{1}{c}\left((\omega_0^2 - \omega^2)^2 + \left(\frac{2c\omega}{\delta}\right)^2\right)^{\frac{1}{4}} \cos\frac{\beta}{2} \tag{25a}$$

$$= \frac{1}{c\sqrt{2}}\sqrt{\sqrt{(\omega_0^2 - \omega^2)^2 + \left(\frac{2c\omega}{\delta}\right)^2} + (\omega_0^2 - \omega^2)} \tag{25b}$$

(a)

(b)

(c)

Figure 1. Penetration depth d as a function of δ/l and ω/ω_0. (a)Penetration depth normalized to l. By normalizing d to l we can clearly see that in the limit of low frequency, d approach l. (b)Penetration depth normalized to δ. By normalizing d to δ we can clearly see that in the limit of high frequency, d approach δ. (c) Illustration of penetration of electromagnetic wave.

Electromagnetic penetration depth

We note that Eq. (23b) and (25b) are exactly the same. Because of that the penetration depth d can be summarize in one expression

$$d = \frac{l\sqrt{2}}{\sqrt{\sqrt{\left(\frac{\omega^2}{\omega_0^2}-1\right)^2 + \left(2\frac{l}{\delta}\frac{\omega}{\omega_0}\right)^2} - \left(\frac{\omega^2}{\omega_0^2}-1\right)}} = \frac{\delta}{\sqrt{2}}\sqrt{\sqrt{\left(1-\frac{\omega_0^2}{\omega^2}\right)^2 + \left(2\frac{l}{\delta}\frac{\omega_0}{\omega}\right)^2} + \left(1-\frac{\omega_0^2}{\omega^2}\right)} \quad (26)$$

Using Eq. (26), we can illustrate penetration depth d as a function of δ/l and ω/ω_0. In Fig. 1a and 1b, d is normalized to l and δ, respectively. We can clearly see that our result converges to the following well-known limits.

$$\lim_{\omega \ll \omega_0} d = l \quad (27a)$$

$$\lim_{\omega \gg \omega_0} d = \delta \quad (27b)$$

Furthermore, we can see that when $\delta = l$, $d \approx l = \delta$.

CONCLUSION

To summarize, we study the intermediate regime between two well-studied limits: high frequency electromagnetic field penetrating a metal with small conductivity and a static field penetrating a superconductor. The former limit can be derived by consider Maxwell equations with Ohmic current. On the other hand, the later limit is obtained by focusing on the diamagnetic current.

By setting the electric current density as the total sum of both Ohmic and Diamagnetic currents, we derive the penetration depth in the intermediate regime. The penetration depth that include both contribution of Ohmic and diamagnetic currents depends on the ratio of δ/l and ω/ω_0 (see Eq. (26)). Figure 1 show that our result converges to the following well-known limits. When $\delta = l$, the penetration depth does not depend strongly to frequency.

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