Another Antimagic Decomposition of Generalized Peterzen Graph

Nur Inayah*, M. Ivran Septiar Musti, and Soffi Nur Masyithoh
Department of Mathematics, UIIN Syarif Hidayatullah Jakarta
Jln. Ir. H. Juanda no.95–Ciputat 15412, Tangerang Selatan, Indonesia
Email: *nur.inayah@uinjkt.ac.id, {irvanseptiar, soffilee0113}@gmail.com

Abstract
A decomposition of a graph \( P \) into a family \( Q \) consisting of isomorphic copies of a graph \( Q \) is \((a, b)\)-\(Q\)-antimagic if there is a bijection \( \varphi : V(P) \cup E(P) \rightarrow \{1, 2, 3, 4, \ldots, v_P + e_P\} \) such that for all subgraphs \( Q' \) isomorphic to \( Q \), the \( Q \)-weights

\[
\varphi(Q') = \sum_{v \in V(Q')} \varphi(v) + \sum_{e \in E(Q')} \varphi(e)
\]

constitute an arithmetic progression \( a, a + b, a + 2b, \ldots, a + (r - 1)b \) where \( a \) and \( b \) are positive integers and \( r \) is the number of subgraphs of \( P \) isomorphic to \( Q \). In this article we prove the existence of a \((a, b)\)-\(P_4\)-antimagic decomposition of a generalized Peterzen graph \( GPz(n, 3) \) for several values of \( b \).

Keywords: covering; decomposition; antimagic; generalized Peterzen.

1. INTRODUCTION

All graphs are finite and simple graphs. The edge and vertex sets of a graph \( P \) are denoted by \( V(P) \) and \( E(P) \), respectively, where \( |V(P)| = v_P \) and \( |E(P)| = e_P \). A graph labeling of a graph \( P \) is a bijective function that carries a set of elements of \( P \) onto a set of labels, usually, a set of positive integers. If the domain and co-domain of this function are \( V(P) \cup E(P) \) and the set \( \{1, 2, 3, 4, \ldots, v_P + e_P\} \), respectively, then it is called a total labeling.

An edge-covering of \( P \) is a family of subgraphs \( Q = \{Q_1, Q_2, Q_3, \ldots, Q_k\} \) such that each edge of \( E(P) \) belongs to at least one of the subgraphs \( Q_i, 1 \leq i \leq k \). Then it is said that \( P \) admits an \((Q_1, Q_2, Q_3, \ldots, Q_k)\)-(edge) covering. If every \( Q_i \) is isomorphic to a given graph \( Q \), then \( P \) admits an \( Q \)-covering [1]. An \((Q_1, Q_2, Q_3, \ldots, Q_k)\)-(edge) covering of \( P \) is called an \((Q_1, Q_2, Q_3, \ldots, Q_k)\)-
decomposition, if \( E(Q_i) \cap E(Q_j) = \emptyset \) for \( i \neq j \). If every \( Q_i \) is isomorphic to a given graph \( Q \), then \( P \) admits an \( Q \)-decomposition.

Suppose \( P \) admits an \( Q \)-decomposition and let \( f : V(P) \cup E(P) \rightarrow \{1, 2, 3, 4 \ldots, v_P + e_P\} \) be a total labeling. An \( Q \)-weight of a subgraph \( Q \) of \( P \) under a total labeling is the sum of all edge and vertex labels on \( Q \). If every subgraph \( Q \in \mathcal{Q} \) has the same \( Q \)-weights, then it is called an \( Q \)-magic decomposition of \( P \). If all \( Q \in \mathcal{Q} \) have distinct \( Q \)-weights, then it is called an \( Q \)-antimagic decomposition of \( P \). In particular, if \( Q \)-weights of all \( Q \in \mathcal{Q} \) are an arithmetic sequence with the first term \( a \) and a common difference \( b \) then it is called an \((a, b)-H \)-antimagic decomposition of \( P \).

Inayah et al. [2] [3] introduced an \((a, b)-Q\)-antimagic total labeling of a graph \( P \) admitting an \( Q \)-decomposition, denoted as an \((a, b)-Q\)-antimagic decomposition as a bijective function \( \phi: V(P) \cup E(P) \rightarrow \{1, 2, 3, 4 \ldots, |V(P)| + |E(P)|\} \) such that for a subgraph \( Q' \) isomorphic to \( Q \), the \( Q \)-weights \( \phi(Q') = \sum_{v \in V(Q')} \phi(v) + \sum_{e \in E(Q')} \phi(e) \) constitute an arithmetic progression \( a, a+b, a+2b, \ldots, a+(r-1)b \) where \( a \) and \( b \) are positive integers and \( r \) is the number of all subgraphs of \( P \) isomorphic to \( Q \). The recent results on this subject can be seen, as an example, in [4] and [5]. The complete results can be seen in a dynamic survey of graph labelings by Gallian [6].

In this article, we proved \((a, b)-P_4\)-antimagic decompositions of generalized Peterzen graphs \( GP_2(n, 3) \). We show that the graphs admit \((a, b)-P_4\)-antimagic decompositions for several values of \( b \).

2. MAIN RESULTS

In this section, we prove the existence of the \((a, b)-P_4\)-antimagic decomposition of the generalized Peterzen graph \( GP_2(n, 3) \) for \( b \in \{1, 2, 3, 4, 5\} \). Watkins [7] defined the generalized Peterzen graph \( GP_2(n, 3) \) as a graph having vertex set

\[
V(GP_2(n, 3)) = \{v_i, u_i: 0 \leq i \leq n-1\}
\]

and edge set

Outer Rim \( E_O((GP_2(n,k)) = \{u_i u_{(i+1) \mod n}\}_{i=0}^{n-1} \),

Inner Rim \( E_I((GP_2(n,k)) = \{v_i v_{(i+k) \mod n}\}_{i=0}^{n-1} \),

Spoke \( E_S(n,k) = \{u_i u_{j}\}_{i=0}^{n} \).

Let \( \mathcal{Q} = \{P_4^0, P_4^1, \ldots, P_4^{n-1}\} \), where the edge and vertex sets of the subgraph \( P_4^i \) defined as follows: For \( i \in [0, n-1] \),

\[
V(P_4^i) = \{v_i, v_{(i+3) \mod n}, u_i, u_{(i+1) \mod n}: 0 \leq i \leq n-1\},
\]

\[
E(P_4^i) = \{v_i v_{(i+3) \mod n}, u_i u_{i+1} u_{(i+1) \mod n}: 0 \leq i \leq n-1\}.
\]

It is not difficult to see that \( \mathcal{Q} = \{P_4^0, P_4^1, \ldots, P_4^{n-1}\} \) is a \( P_4 \)-decomposition of \( GP_2(n, 3) \). Figure 1 displays the generalized Peterzen Graph \( GP_2(n, 3) \).
Figure 1. Generalized Peterzen Graph $GP_z(n, 3)$

**Theorem 1.** For any integer $n \geq 7$, the graph $GP_z(n, 3)$ has a $(20n + 4, 1) P_4$-antimagic decomposition.

**Proof.** Define a total labeling $\psi_q$ on the edges and vertices of the graph $GP_z(n, 3)$ in the following way

\[
\begin{align*}
\psi_q(v_{i}v_{(i+3) \mod n}) &= \begin{cases} 
 i + 5 & \text{for } i \in [0, n - 5] \\
 -n + i + 5 & \text{for } i \in [n - 4, n - 1] 
\end{cases} \\
\psi_q(v_{i}u_{i}) &= \begin{cases} 
 2n & \text{for } i = 0 \\
 n + i & \text{for } i \in [1, n - 1] 
\end{cases} \\
\psi_q(u_{i}u_{(i+1) \mod n}) &= \begin{cases} 
 2n + 1 & \text{for } i = 0 \\
 3n - i + 1 & \text{for } i \in [1, n - 1] 
\end{cases} \\
\psi_q(u_{i}) &= \begin{cases} 
 4n - i - 3 & \text{for } i \in [0, n - 5] \\
 5n - i - 3 & \text{for } i \in [n - 4, n - 1] \\
 4n + i + 1 & \text{for } i \in [0, n - 1] 
\end{cases}
\end{align*}
\]

It can be seen that the labeling $\psi_q$ is a bijective function from $E(GP_z(n, 3)) \cup V(GP_z(n, 3))$ to \{1, 2, 3, 4, ..., 3n\} and $\psi_q(V(GP_z(n, 3))) = \{1, 2, 3, 4 ..., n + 1\}$. Furthermore, the $P_4$-weight under the labeling $\psi_q$ are as follows.

\[
w(P_4^i) = \begin{cases} 
\psi_q(v_{(i+3)}) + \psi_q(v_{i}v_{(i+3)}) + \psi_q(v_{i}) + \psi_q(v_{i}u_{i}) + \psi_q(u_{i}) \\
+ \psi_q(u_{i}u_{(i+1)}) + \psi_q(u_{(i+1)}), & \text{for } i \in [0, n - 4] \\
\psi_q(v_0) + \psi_q(v_{i}v_0) + \psi_q(v_{i}) + \psi_q(v_{i}u_{i}) + \psi_q(u_{i}) \\
+ \psi_q(u_{i}u_{(i+1)}) + \psi_q(u_{(i+1)}), & \text{for } i = [n - 3] \\
\psi_q(v_{i}) + \psi_q(v_{i}v_1) + \psi_q(v_{i}) + \psi_q(v_{i}u_{i}) + \psi_q(u_{i}) \\
+ f_q(u_{i}u_{(i+1)}) + f_q(u_{(i+1)}), & \text{for } i = [n - 2] \\
\psi_q(v_2) + \psi_q(v_{i}v_2) + \psi_q(v_{i}) + \psi_q(v_{i}u_{i}) + \psi_q(u_{i}) \\
+ \psi_q(u_{i}u_1) + \psi_q(u_0), & \text{for } i = [n - 1] 
\end{cases}
\]

For $i \in [0, n - 1]$, under labeling $\psi_q$, we find
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\[ w(P^i_4) = w(P^i_4) = \psi_q(v_{i+3}) + \psi_q(v_i v_{i+3}) + \psi_q(v_i) + \psi_q(u_i) + \psi_q(u_{i+1}) = (4n + (i + 3) + 1) + (i + 5) + (4n + i + 1) + (2n) + (4n - i - 3) + (2n + 1) + (4n - (i + 1) - 3) = 20n + i + 4 \]

Since \[ w(P^{i+1}_4) - w(P^i_4) = 1 \] and \[ w(P^0_4) = 20n + 4 \], the generalized Peterzen \( GPz(n, 3) \) admits a \((20n + 4, 1)\)-\(P_4\)-antimagic decomposition.■

Figure 2. A \((144, 1) - P_4\)-Antimagic Decomposition of Generalized Peterzen Graph \( GPz(7, 3) \) (left), a \((164, 1) - P_4\)-Antimagic Decomposition of Generalized Peterzen Graph \( GPz(8, 3) \) (right).

Theorem 2. For any integer \( n \geq 7 \), the graph \( GPz(n, 3) \) has a \((14n + 4, 2)\)-\(P_4\)-antimagic decomposition.

Proof. Define a total labeling \( \psi_e \) on the edges and vertices of the graph \( GPz(n, 3) \) in the following way

\[
\begin{align*}
\psi_e(v_{i} v_{(i+3) \mod n}) &= 4n + i + 1 & \text{for } i \in [0, n - 1] \\
\psi_e(v_{i} u_{i}) &= \begin{cases} 
4n + i - 2 & \text{for } i \in [0, 2] \\
3n + i - 2 & \text{for } i \in [3, n - 1] 
\end{cases} \\
\psi_e(u_{i} u_{(i+1) \mod n}) &= \begin{cases} 
3n - 2i - 2 & \text{for } i \in [0, n - 2] \\
5n - 2i - 2 & \text{for } i = [n - 1] 
\end{cases} \\
\psi_e(u_{i}) &= n + 2i + 1 & \text{for } i \in [0, n - 1] \\
\psi_e(v_{i}) &= \begin{cases} 
-i + 3 & \text{for } i \in [0, 2] \\
-n + 3 & \text{for } i \in [3, n - 1] 
\end{cases}
\end{align*}
\]

It can be seen that the labeling \( \psi_e \) is a bijective function from \( E(GPz(n, 3)) \cup V(GPz(n, 3)) \) to \( \{1, 2, 3, 4, \ldots, 3n\} \) and \( \psi_e(V(GPz(n, 3))) = \{1, 2, 3, 4, \ldots, n + 1\} \). Furthermore, the \( P_4 \)-weight under the labeling \( \psi_e \) are as follows
For $i \in [0, n - 1]$, under labeling $\psi_e$, we find

$$w(p^i_4) = \begin{cases} 
\psi_e(v_{(i+3)}) + \psi_e(v_i v_{(i+3)}) + \psi_e(v_i) + \psi_e(v_i u_i) + \psi_e(u_i) + \psi_e(u_i u_{(i+1)}) + \psi_e(u_{(i+1)}), & \text{for } i \in [0, n - 4] \\
\psi_e(v_0) + \psi_e(v_i v_0) + \psi_e(v_i) + \psi_e(v_i u_i) + \psi_e(u_i) + \psi_e(u_i u_{(i+1)}) + \psi_e(u_{(i+1)}), & \text{for } i = [n - 3] \\
\psi_e(v_1) + \psi_e(v_i v_1) + \psi_e(v_i) + \psi_e(v_i u_i) + \psi_e(u_i) + \psi_e(u_i u_{(i+1)}) + \psi_e(u_{(i+1)}), & \text{for } i = [n - 2] \\
\psi_e(v_2) + \psi_e(v_i v_2) + \psi_e(v_i) + \psi_e(v_i u_i) + \psi_e(u_i) + \psi_e(u_i u_{(i+1)}) + \psi_e(u_{(i+1)}), & \text{for } i = [n - 1] 
\end{cases}$$

Since $w(p^{i+1}_4) - w(p^i_4) = 2$ and $w(p^0_4) = 14n + 4$, the generalized Peterzen GPZ$(n, 3)$ admits a $(14n + 4, 2)$-$P_4$- antimagic decomposition.

**Theorem 3.** For any integer $n \geq 7$, the graph GPZ$(n, 3)$ has a $(19n + 5, 3)$-$P_4$- antimagic decomposition.  

**Proof.** Define a total labeling $\Psi_r$ on the edges and vertices of the graph GPZ$(n, 3)$ in the following way

$$
\begin{align*}
\psi_r(v_{i} v_{(i+3) \mod n}) &= i + 1 & \text{for } i \in [0, n - 1], \\
\psi_r(v_i u_i) &= 2n + i - 1 & \text{for } i \in [0, 1], \\
\psi_r(u_i u_{(i+1) \mod n}) &= n + i - 1 & \text{for } i \in [2, n - 1], \\
\psi_r(u_i) &= 2n + i + 1 & \text{for } i \in [0, n - 1], \\
\psi_r(u_{(i+1)}) &= 3n - i + 3 & \text{for } i \in [0, 2], \\
\psi_r(v_i) &= 4n - i + 3 & \text{for } i \in [3, n - 1], \\
\psi_r(v_0) &= 5n + i - 2 & \text{for } i \in [0, 2], \\
\psi_r(v_3) &= 4n + i - 2 & \text{for } i \in [3, n - 1].
\end{align*}
$$

It can be seen that the labeling $\psi_r$ is a bijective function from $E(GPZ(n, 3)) \cup V(GPZ(n, 3))$ to $\{1, 2, 3, 4, ..., 3n\}$ and $\psi_r(V(GPZ(n, 3))) = \{1, 2, 3, 4, ..., n + 1\}$. Furthermore, the $P_4$-weight under the labeling $\psi_r$ are as follows
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For $i \in [0, n - 1]$, under labeling $\psi_r$, we find

\[
w(P^i_4) = \begin{cases} 
\psi_r(v_{i+3}) + \psi_r(v_i v_{i+3}) + \psi_r(v_i) + \psi_r(v_iu_i) + \psi_r(u_i) \\
+ \psi_r(u_iu_{i+1}) + \psi_r(u_{i+1}), & \text{for } i \in [0, n - 4] \\
\psi_r(v_0) + \psi_r(v_i v_0) + \psi_r(v_i) + \psi_r(v_iu_i) + \psi_r(u_i) \\
+ \psi_r(u_iu_{i+1}) + \psi_r(u_{i+1}), & \text{for } i = [n - 3] \\
\psi_r(v_1) + \psi_r(v_i v_1) + \psi_r(v_i) + \psi_r(v_iu_i) + \psi_r(u_i) \\
+ \psi_r(u_iu_{i+1}) + \psi_r(u_{i+1}), & \text{for } i = [n - 2] \\
\psi_r(v_2) + \psi_r(v_i v_2) + \psi_r(v_i) + \psi_r(v_iu_i) + \psi_r(u_i) \\
+ \psi_r(u_iu_{i+1}) + \psi_r(u_0), & \text{for } i = [n - 1]
\end{cases}
\]

Since $w(P^i_{4+1}) - w(P^i_4) = 3$ and $w(P^0_4) = 19n + 5$, the generalized Peterzen GPz(n, 3) admits a $(19n + 5, 3) - P_4$-antimagic decomposition. }

**Theorem 4.** For any integer $n \geq 7$, the graph GPz(n, 3) has a $(13n + 5, 4) - P_4$-antimagic decomposition.

**Proof.** Define a total labeling $\Psi_t$ on the edges and vertices of the graph GPz(n, 3) in the following way

\[
\Psi_t(v_i v_{i+3} \text{ mod n}) = \begin{cases} 
4n - i + 1 & \text{for } i = 0 \\
5n - i + 1 & \text{for } i \in [1, n - 1] \\
4n + i - 3 & \text{for } i \in [0, 3] \\
3n + i - 3 & \text{for } i \in [4, n - 1]
\end{cases}
\]

\[
\Psi_t(v_i u_i) = \begin{cases} 
3n + i - 3 & \text{for } i \in [4, n - 1]
\end{cases}
\]

\[
\Psi_t(u_i u_{i+1} \text{ mod n}) = \begin{cases} 
n + 2 + 2i & \text{for } i \in [0, n - 1]
\end{cases}
\]

\[
\Psi_t(u_i) = \begin{cases} 
3n + 2i - 1 & \text{for } i = 0 \\
n + 2i - 1 & \text{for } i \in [1, n - 1]
\end{cases}
\]

\[
\Psi_t(v_i) = \begin{cases} 
4 - i & \text{for } i \in [0, 3] \\
n + 4 - i & \text{for } i \in [4, n - 1]
\end{cases}
\]

It can be seen that the labeling $\Psi_t$ is a bijective function from $E(GPz(n, 3)) \cup V(GPz(n, 3))$ to \{1, 2, 3, 4, ..., 3n\} and $\Psi_t(V(GPz(n, 3))) = \{1, 2, 3, 4, ..., n + 1\}$. Furthermore, the $P_4$-weight under the labeling $\Psi_t$ are as follows
Theorem 5. For any integer \( n \geq 7 \), the graph \( GPz(n, 3) \) has a \((18n + 6, 5)\)-\(P_4\)-antimagic decomposition.

Proof. Define a total labeling \( \Psi_y \) on the edges and vertices of the graph \( GPz(n, 3) \) in the following way

\[
\begin{align*}
\psi_y(v_i v_{i+3} \mod n) &= 2i + 1 & \text{for } i \in [0, n - 1] \\
\psi_y(v_i u_i) &= 2i + 2 & \text{for } i \in [0, n - 1] \\
\psi_y(u_i u_{i+1} \mod n) &= \begin{cases} 3n + i - 3 & \text{for } i \in [0, 3] \\ 2n + i - 3 & \text{for } i \in [4, n - 1] \\ \end{cases} \\
\psi_y(u_i) &= \begin{cases} 4n + i & \text{for } i = 0 \\ 3n + i & \text{for } i \in [1, n - 1] \end{cases} \\
\psi_y(v_i) &= \begin{cases} 4n - i + 4 & \text{for } i \in [0, 3] \\ 5n - i + 4 & \text{for } i \in [4, n - 1] \end{cases}
\end{align*}
\]

It can be seen that the labeling \( \psi_y \) is a bijective function from \( E(GPz(n, 3)) \cup V(GPz(n, 3)) \) to \( \{1, 2, 3, 4, ..., 3n\} \) and \( \psi_y(V(GPz(n, 3))) = \{1, 2, 3, 4, ..., n + 1\} \). Furthermore, the \( P_4 \)-weight under the labeling \( \psi_y \) are as follows.
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For every integer $i \geq 0$, and odd positive integers $w = 18i + 6$, the generalized Peterzen $GPz(n, 3)$ admits a $(18n + 6, i, 3)$-$P_{4}$-antimagic decomposition.■

3. CONCLUSION

In this article, we proved the existence of $(a, b)$-$P_{4}$-antimagic decompositions of the generalized Peterzen graph $GPz(n, 3)$ for (i) every integer $n \geq 7$ and odd positive integers $b \in \{1, 3, 5\}$; and (ii) every integer $n \geq 7$ and even positive integers $b \in \{2, 4\}$.

The open problems related to these results are as follows:

For every integer $6 \geq n$ and positive integers $b$, find $(a, b)$-$P_{4}$-antimagic decompositions of the generalized Peterzen graph $GPz(n, 3)$.

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