Some Notes on Relative Commutators

Masoumeh Ganjali and Ahmad Efranian*
Department of Pure Mathematics and Center of Excellence in Analysis on Algebraic Structures, Fedowsi University of Mashhad, P.O.Box 1159-91775 Mashhad, Iran
Email: m.ganjali20@yahoo.com, *erfanian@math.um.ac.ir

Abstract
Let $G$ be a group and $\alpha \in Aut(G)$. An $\alpha$-commutator of elements $x, y \in G$ is defined as $[x, y]_\alpha = x^{-1}y^{-1}xy^\alpha$. In 2015, Barzegar et al. introduced an $\alpha$-commutator of elements of $G$ and defined a new generalization of nilpotent groups by using the definition of $\alpha$-commutators which is called an $\alpha$-nilpotent group. They also introduced an $\alpha$-commutator subgroup of $G$, denoted by $D_\alpha(G)$ which is a subgroup generated by all $\alpha$-commutators. In 2016, an $\alpha$-perfect group, a group that is equal to its $\alpha$-commutator subgroup, was introduced by authors of this paper and the properties of such group was investigated. They proved some results on $\alpha$-perfect abelian groups and showed that a cyclic group $G$ of even order is not $\alpha$-perfect for any $\alpha \in Aut(G)$. In this paper, we may continue our investigation on $\alpha$-perfect groups and in addition to studying the relative perfectness of some classes of finite $p$-groups, we provide an example of a non-abelian $\alpha$-perfect 2-group.

Keywords: Auto-commutator subgroup; finite $p$-group; normal subgroup; perfect group.

Misalkan $G$ grup dan $\alpha \in Aut(G)$. Suatu $\alpha$-komutator dari unsur-unsur $x, y \in G$ didefinisikan sebagai $[x, y]_\alpha = x^{-1}y^{-1}xy^\alpha$. Pada tahun 2015, Barzegar et al. memperkenalkan $\alpha$-komutator dari unsur-unsur di $G$ dan mendefinisikan sebuah perumuman baru dari grup-grup nilpoten dengan menggunakan definisi dari $\alpha$-komutator yang dinamakan grup $\alpha$-nilpoten. Mereka juga memperkenalkan suatu subgrup $\alpha$-komutator dari $G$ yang dilambangkan dengan $D_\alpha(G)$ yang merupakan subgrup yang dibangun dari semua $\alpha$-komutator. Pada tahun 2016, grup $\alpha$- sempurna, yaitu grup yang subgrup $\alpha$-komutatornya sama dengan grup itu sendiri, diperkenalkan oleh penulis paper ini dan sifat-sifat grup tersebut juga diselidiki. Mereka membuktikan beberapa sifat dari grup abel $\alpha$- sempurna dan memperlihatkan bahwa suatu grup siklis $G$ dengan order genap bukan grup $\alpha$-sempurna untuk setiap $\alpha \in Aut(G)$. Di paper ini kita akan melanjutkan investigasi kita pada grup-grup $\alpha$-sempurna dan sebagai tambahan dalam mempelajari ke sempurnaan relatif dari kelas-kelas dari $p$-grup berhingga, kita akan melihat contoh dari 2-grup $\alpha$-sempurna yang non abel.

Kata kunci: subgrup auto-komutator; $p$-grup berhingga; subgrup normal; grup sempurna.

Mathematics Subject Classification (2010): Primary 20F12; Secondary 20D45.
1. INTRODUCTION

In 1994, an auto-commutator \([x, \alpha] = x^{-1} x^\alpha\) of elements \(x \in G\) and \(\alpha \in \text{Aut}(G)\) was introduced by Hegarty, [1]. If \(\alpha_g\) is an inner automorphism such that \(x^{\alpha_g} = g^{-1} x g\) then auto-commutator \([x, \alpha_g] = x^{-1} g^{-1} x g\) is the ordinary commutator of two elements \(x, g \in G\). Hegarty generalized the definition of the center of \(G\), \(Z(G) = \{x \in G : x^\alpha = x, \forall y \in G\}\) to the absolute center \(L(G) = \{x \in G : x^\alpha = x, \forall \alpha \in \text{Aut}(G)\}\) of \(G\). One can check that \(L(G)\) is an invariant subgroup of \(G\) which is contained in \(Z(G)\). He also introduced the auto-commutator subgroup of \(G\), denoted by \(K(G)\), which is a characteristic subgroup generated by all auto-commutators. Clearly, the commutator subgroup \(G'\) is contained in \(K(G)\). Investigation of the relative commutators are interesting for some authors, for instance Barzegar et al. [2] also introduced a new generalization of commutators with respect to a fixed automorphism of group \(G\). Let \(\alpha \in \text{Aut}(G)\), then an \(\alpha\)-commutator of two elements \(x, g \in G\) is defined as \([x, y]_{\alpha} = x^{-1} y^\alpha x y\) which is equal to the ordinary commutator \([x, y] = x^{-1} y x y\) whenever \(\alpha\) is the identity automorphism. In [2], the subgroup which is generated by all \(\alpha\)-commutators was denoted by \(D_\alpha(G)\) and called \(\alpha\)-commutator subgroup of \(G\). It is not difficult to prove that \(D_\alpha(G)\) is a normal subgroup of \(G\) that is contained in \(K(G)\). Authors of [2] also introduced a new generalization of a nilpotent group \(G\), which is called an \(\alpha\)-nilpotent group for a fixed automorphism \(\alpha\) of \(G\). Here, we may present the definition of an \(\alpha\)-nilpotent group \(G\). We start by the definition of a lower central \(\alpha\)-series. Put \(\Gamma^\alpha_0(G) = G\) and \(\Gamma^\alpha_1(G) = D_\alpha(G)\) and define inductively \(\Gamma^\alpha_{n+1}(G) = [G, \Gamma^\alpha_n(G)]_{\alpha} = ([x, y]_{\alpha} : x \in G, y \in \Gamma^\alpha_n(G))\), \(n \geq 1\). We can see that \(\Gamma^\alpha_n(G)\) is a normal subgroup of \(G\) which is invariant under \(\alpha\) and \(\Gamma^\alpha_{n+1}(G) \leq \Gamma^\alpha_n(G)\), for all \(n \geq 1\). Following normal series is called a lower central \(\alpha\)-series \(G \geq \Gamma^\alpha_2(G) \geq \cdots \geq \Gamma^\alpha_n(G) \geq \cdots\).

A group \(G\) is called an \(\alpha\)-nilpotent group of nilpotency class \(n\) if \(\Gamma^\alpha_n(G) = \{1\}\) and \(\Gamma^\alpha_{n+1}(G) \neq \{1\}\). Clearly, if \(\alpha\) is considered as the identity automorphism, then an \(\alpha\)-nilpotent group is the ordinary one. In [2], it was proved that an \(\alpha\)-nilpotent group is nilpotent, but the converse is not valid in general. For instance, authors proved that the cyclic group of order \(n = p_1 p_2 \cdots p_t\) is \(\alpha\)-nilpotent if and only if \(\alpha\) is the identity automorphism, for distinct primes \(p_1, p_2, \ldots, p_t\). Authors of [3] continued investigation on \(\alpha\)-nilpotent groups and proved some new results on this new concept. For example, they proved that an extra special \(p\)-group, \(p\) is an odd prime number, is nilpotent with respect to a non-identity automorphism \(\alpha\) but is not nilpotent relative to all its automorphisms. For an inner automorphism \(\alpha_g \in \text{Inn}(G)\), we can see that nilpotency and \(\alpha_g\)-nilpotency are equivalent. Therefore, we may ask the following question.

**Question.** Is there a non-inner automorphism \(\alpha\) of nilpotent group \(G\) such that \(G\) is \(\alpha\)-nilpotent?
Some Notes on Relative Commutators

This question was answered for finitely generated abelian groups, for more details see [3]. Actually, authors classified all finitely generated abelian groups which are nilpotent with respect to a non-inner automorphism. Furthermore, they proved some results on relative normal and absolute normal subgroups of some classes of finite groups. In [4], they introduced an $\alpha$-perfect group $G$, a group which is equal to its $\alpha$-commutator subgroup, for a fixed automorphism $\alpha$ of $G$. If $G'$ is the ordinary commutator subgroup of $G$, then $G' \leq D_\alpha(G)$ for all $\alpha \in \text{Aut}(G)$. It follows that if $G$ is a perfect group, then it is perfect with respect to all its automorphisms. One can check that an $\alpha$-nilpotent group cannot be $\alpha$-perfect, but the symmetric group of order $n!$, $S_n$ is an example of a non-nilpotent group where is not $\alpha$-perfect, because $D_\alpha(S_n) = (S_n) = A_n$ for all $\alpha \in \text{Aut}(G)$. The relative perfectness of abelian groups was studied by authors of [4]. In this paper, we may continue our investigation on relative perfect groups and prove some new results on some classes of finite non-abelian $p$-groups.

2. RELATIVE PERFECT GROUPS

In this section, we recall the definition of an $\alpha$-perfect group for a fixed automorphism $\alpha$. At first, we present some results on relative perfect groups that were proved in [4]. Finally, we may add some new results on non-abelian relative perfect groups.

Definition 2.1. Let $G$ be a group and $\alpha \in \text{Aut}(G)$. A group $G$ is called an $\alpha$-perfect group, whenever $G = D_\alpha(G)$.

Definition 2.2. If $G$ is a finite group and $\alpha \in \text{Aut}(G)$, then a subgroup $H$ of $G$ is called an $\alpha$-normal subgroup of $G$, denoted by $H \trianglelefteq^\alpha G$, if $g^{-1}h^\alpha g \in H$ for all $g \in G$ and $h \in H$. If $H$ is $\alpha$-normal with respect to all automorphisms $\alpha \in \text{Aut}(G)$, then $G$ is called an absolute normal subgroup of $G$.

Lemma 2.3. ([4]) Let $H$ be a subgroup of finite group $G$, then (i) if there exists an $\alpha \in \text{Aut}(G)$ such that $H \trianglelefteq^\alpha G$, then $H$ is a normal subgroup of $G$, (ii) $H$ is an absolute normal subgroup of $G$ if and only if $K(G) \leq H$.

It might be important to find all proper absolute normal subgroups of given finite group $G$. In [3] and [4], the structure of absolute normal subgroups of some classes of finite groups were given. For instance, we have the following results.

Lemma 2.4. ([4]) If $G \cong \mathbb{Z}_{2^n}^m$ such that $(2, m) = 1$, then the proper subgroup $H$ of $G$ is absolute normal if and only if $H = 2G$.

Theorem 2.5. ([3]) (i) If $D_{2n} = \langle x, y : x^n = y^2 = 1, yxy = x^{-1} \rangle$, then $\langle x \rangle$ is the only proper absolute normal subgroup of $D_{2n}$. (ii) Semi-dihedral 2-group $SD_{2n} = \langle x, y : x^n = y^2 = 1, yxy = x^{-1} = x^{2n+1} \rangle, n \geq 3$ has the only proper absolute normal subgroups given by $\langle x \rangle, \langle x^2 \rangle, \langle x^2, y \rangle, \langle x^2, yx \rangle$. (iii) Generalized
quaternion 2-group \( Q_{2^n} = \langle x, y : x^{2^n} = 1, x^2 = y^2, yxy^{-1} = x^{-1} \rangle, n \geq 3 \) has the only proper absolute normal subgroup \( \langle x \rangle \).

(iv) Twisted dihedral 2-group \( S, D_{2^n} = \langle x, y : x^{2^n} = y^2 = 1, yxy^{-1} = x^{2^n+1} \rangle, n \geq 3 \) has the only proper absolute normal subgroup \( \langle x^2, y \rangle \).

**Theorem 2.6.** \([3]\) If \( p \) is an odd prime number and \( M_n(p) = \langle x, y : x^p = 1, xy = yx^{p-1} \rangle, n \geq 3 \), then \( M_n(p) \) does not possess a proper absolute normal subgroup.

Next lemma, talks about the existence of an \( \alpha \)-normal subgroup in abelian \( \alpha \)-perfect group \( G \).

**Lemma 2.7.** \([4]\) Let \( G \) be a finite abelian group. Then \( G \) is \( \alpha \)-perfect if and only if \( G \) does not possess a proper \( \alpha \)-normal subgroup.

If \( G \) is a finite cyclic group of order \( n \), then \( \alpha \in Aut(G) \) if and only if \( x^\alpha = ux \) such that \((u, n) = 1\), for all \( x \in G \). We denote such \( \alpha \) by \( \alpha_u \).

**Lemma 2.8.** \([4]\) Let \( G \) be a cyclic group of order \( n \) and \( \alpha_u \in Aut(G) \) be a non-identity automorphism. Then \( G \) is \( \alpha_u \)-perfect if only if \((u - 1, n) = 1\).

By Lemma 2.8, we can conclude that there is no \( \alpha \)-perfect cyclic group of even order, for all automorphisms \( \alpha \) of such group. If \( p \) is an odd prime number, then \( Z_{p^r}, r > 1 \), is \( \alpha_u \)-perfect for each \( 1 < u < p \), but it is not \( \alpha_{p+1} \)-perfect.

Now, we are ready to prove some new results on relative perfect groups.

**Lemma 2.9.** If \( \alpha_g \) is an inner automorphism and \( \beta = \alpha \circ \alpha_g \), then \( G \) is \( \alpha \)-perfect if and only if \( \beta \)-perfect.

**Proof.** We can see that \([x, y]_\beta = [x, y]_\alpha [y, g]_\alpha \) and since \([y, g] \in G' \) \( \leq D_\alpha(G) \) and \( D_\alpha(G) \) is \( \alpha \)-invariant, then \([x, y]_\beta \in D_\alpha(G) \) and so \( D_\beta(G) \leq D_\alpha(G) \). We can write \( \alpha = \beta \circ \alpha_g^{-1} \) and prove \( D_\alpha(G) \leq D_\beta(G) \). Now, we are done. \( \blacksquare \)

**Example 2.10.** (i) If \( G \) is isomorphic to one of the groups where are defined in Theorem 2.5, then \( G \) possesses a proper absolute normal subgroup. Furthermore, we know that \( D_\alpha(G) \leq K(G) \), for all \( \alpha \in Aut(G) \). So by Lemma 2.3, \( D_\alpha(G) \) is a proper subgroup of \( G \) and \( G \) is not \( \alpha \)-perfect for any \( \alpha \in Aut(G) \). (ii) Assume that that \( Q_8 = \langle xy : x^4 = 1, x^2 = y^2, yxy^{-1} = x^{-1} \rangle, n \geq 3 \) and \( \alpha \in Aut(Q_8) \) is an automorphism by argument \( x^\alpha = y \) and \( y^\alpha = xy \). Then \( D_\alpha(Q_8) = Q_8 \) and \( Q_8 \) is an \( \alpha \)-perfect group.
By Theorem 2.6, if $G \cong M_n(p)$, then $G$ does not possess any proper absolute normal subgroup. Here, we may prove that $M_n(p)$ is not $\alpha$-perfect for any $\alpha \in Aut(G)$.

**Theorem 2.11.** If $G \cong M_n(p) = \langle x, y : x^{p^n} = y^p = 1, xy = yx^{p^{n-1}} \rangle$ for an odd prime number $p$ and $n \geq 3$, then $G$ is not $\alpha$-perfect for any $\alpha \in Aut(G)$.

**Proof.** We can see that $|G| = p^n$, $Z(G) = \langle x^p \rangle$, $G' = \langle x^{p^{n-2}} \rangle$. The automorphism group of $G$, $Aut(G)$, is equal to $\left\{ \alpha_{ijk} : x^{a_{ijk}} = x^{ij}, y^{a_{ijk}} = x^{kp^{n-2}}, 0 \leq i \leq p^{n-1}, 0 \leq j, k \leq p-1 \right\}$. It is not difficult to see that $[x, x]_{a_{ijk}} = x^{ij}y^j, [x, y]_{a_{ijk}} = x^{(k+1)p^{n-2}}, [y, x]_{a_{ijk}} = x^{kp^{n-2}}x^{ij}y^j, [y, y]_{a_{ijk}} = x^{kp^{n-2}}$.

Therefore, $D_{a_{ijk}}(G) = \langle x^{i-1}y^j, x^{p^{n-2}} \rangle$. If $i \neq 1$, then $|x^{i-1}y^j| = |x|$, also $(i-1)p^{n-2} \equiv p^{n-2} \pmod{p}$, and for a positive integer $m$ we have $x^{(i-1)m(m-1)/2} = x^{m(i-1)(i-3)/2}$.

Now, if we put $m = p^{n-2}$, then since $n \geq 3$, $p^{n-2} \geq p$ and $(x^{i-1}y^j)p^{n-2} = x^{i-1}p^{n-2} = x^{p^{n-2}}$. It means that $x^{p^{n-2}} \in \langle x^{i-1}, y^j \rangle$ and $D_{a_{ijk}}(G) \in \langle x^{i-1}, y^j \rangle$ is a proper subgroup of $G$, because $|x^{i-1}y^j| = p^{n-2}$. Now if, $i = 1$ and $j \neq 0$, then $D_{a_{ijk}}(G) \in \langle y^j, x^{p^{n-2}} \rangle$ and since $|y^j| = |x^{p^{n-2}}| = p$, then $D_{a_{ijk}}(G) \neq G$. In case $j = 0$, we have $D_{a_{ijk}}(G) = G \neq G$. If $i \neq 1$ and $j \equiv 1 \pmod{p}$, then $i = 1 + lp^s$, where $(l, p) = 1, s = 1, 2, \ldots, n-2$. In this case, $G$ is $\alpha_{ijk}$-nilpotent by Theorem 4.10 of [2] and so $D_{a_{ijk}}(G) \neq G$. Hence, we are finished.

**Theorem 2.12.** Let $p$ be an odd prime number. If $G = \langle a, b : a^{p^2} = b^p = 1, a^{-1}b = b^{p+1} \rangle$ is a $p$-group of order $p^5$ and nilpotency class three, then $G$ is not $\alpha$-perfect for any $\alpha \in Aut(G)$.

**Proof.** The automorphism group of $G$ is $Aut(G) = \{ \alpha_{z, \omega, \mu} : a^{z \omega, \mu} = a(a^{-1}b^{\omega})^\mu, b^{\omega, \mu} = a^\mu b^\omega ; \omega \equiv 0, \mu p^2 \equiv 0 \}$. Let $\alpha = \alpha_{z, \omega, \mu} \in Aut(G)$, be an arbitrary automorphism such that $z = pt$ and $\mu = pk$ for some integers $t, k \in \mathbb{Z}$. Then

(i) $[a, a] = (a^{pt}b^\omega)^{pk} = a^{pt}b^{(pt+1)^k-1} = b^{\omega, \mu}$, (ii) $[a, b] = a^{-1}b^{-1}ab^{pt}b^\omega = a^{-1}ab^{-1}a^{-1}b^{-1}a^{pt}b^\omega = b^{pt+1}a^{pt}b^\omega = b^{pt+1}a^{pt}b^\omega$, (iii) $[b, a] = b^{-1}a^{-1}ba(a^{-1}b^{\omega})^{pt} = b^{pt}b^{(pt+1)^k-1}$, and (iv) $[b, b] = b^{-1}a^{pt}b^\omega = a^{pt}b^{-(p+1)^k}$.

69 | InPrime: Indonesian Journal of Pure and Applied Mathematics
Assume that $\alpha \in D_a(G)$, then there exists an integer $j$ such that $a = (a^{p^j}b^{-1})^j$. Put $m = \frac{(p+1)^{\nu} - 1}{(p+1)^{\nu} - 1}$, then since $Z(G) = \langle b^{p^j} \rangle$ we have $a = (a^{p^j}b^{-1})^j b^{-p^j y} = a^{p^j}b^{-m}b^{-p^j y} = a$. We know that $ba = ab^{p+1}$, so we can conclude that $b^{p^j}b^{-m}b^{-p^j y} = a^{p^j}b^{-m}b^{-p^j y}b^{p+1}$, and so $a^{p^j}b^{(p+1)^{\nu}}b^{-m} = a^{p^j}b^{-m}b^{p+1}$ and $b^{(p+1)^{\nu}} = b^{p+1}$. But $b^{(p+1)^{\nu}} = b^{p^j y + 1}$, therefore we have $b^{p^j y + 1} = b^{p+1}$ and $b^{p^j y - p} = 1$ which implies that $p^3 | p^2 t j - p$, a contradiction. Hence $D_a(G) < G$ and $G$ is not $\alpha$-perfect.

In [4], has been shown that for every finite abelian group $G$, there exists a finite abelian group $H$ and $\alpha \in Aut(H)$ such that $D_a(H) \cong G$. Here, we may improve this result to finitely generated abelian groups.

**Proposition 2.13.** If $\alpha \in Aut(G)$ and $\beta \in Aut(H)$, then $D_{\alpha \times \beta}(G \times H) = D_{\alpha}(G) \times D_{\beta}(H)$.

**Proof.** It is straightforward. ■

**Theorem 2.14.** Assume that $G = \prod_{i=1}^{t} Z \times \prod_{i=1}^{t} Z \times \cdots \times \prod_{i=1}^{t} Z \times G_i$ such that $G_i$ is a finite abelian group. Then there exist an abelian group $H$ and $\alpha \in Aut(H)$ such that $D_a(H) \cong G$.

**Proof.** By Theorem 3.7 of [4], for finite group $G_i$, there exist abelian group $H_i$ and $\beta \in Aut(H_i)$ such that $D_{\beta}(H_i) \cong G_i$. Now, if $\alpha \in Aut(Z \times Z)$ by argument $(a, b)^\alpha = (a + b, b)$, then $D_{\alpha}(Z \times Z) = \langle (a, b) : a, b \in Z \rangle \cong Z$, where $Z \cong Z$ for $i = 1, \ldots, t$. Now, it is enough to put $H = \prod_{i=1}^{t} Z \times \cdots \times \prod_{i=1}^{t} Z \times H_1$ and $\alpha = \alpha_1 \times \cdots \times \alpha_t \times \beta$, then $D_{\alpha}(H) = D_{\alpha_1}(Z \times Z) \times D_{\alpha_2}(Z \times Z) \times \cdots \times D_{\alpha_t}(Z \times Z) \times D_{\beta}(H_1) \cong G$, and the proof is completed. ■

**REFERENCES**


